Effect of internal period on the optical dispersion of indefinite-medium materials

Linfang Shen*
Department of Information Science and Electronic Engineering, Electromagnetic Academy, Zhejiang University, Zhejiang Province, Hangzhou 310027, People's Republic of China

Tzong-Jer Yang
Department of Electrophysics, National Chiao Tung University, Hsinchu 30050, Taiwan, Republic of China

Yuan-Fong Chau
Department of Electronic Engineering, Ching Yun University, Jung Li 320, Taiwan, Republic of China
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The effect of internal period on the optical dispersion of indefinite medium material (IDMM) is analytically studied under the condition of the period much smaller than the operating wavelength, based on a simplified dipole model for the material. Interesting phenomena associated with the internal period, such as upper cutoff for wave vector, additional propagating mode, and parabolic dispersion in a limiting case are demonstrated in detail. However, for the normal wave vector (**k**) region, where $|\mathbf{k}|\sim k_0$ or $|\mathbf{k}|\ll k_0$ (k_0 is the free-space wave number), the hyperbolic dispersion behavior can still be realized by IDMM as long as its internal period is small enough. Our analysis also shows that unlike the homogeneous indefinite medium, there exists no special boundary for IDMM, on which the refraction problem cannot be physically solved. Finally, the dispersion properties obtained from the dipole model are verified by using a real example of layered IDMM.

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I. INTRODUCTION

Indefinite media, as a novel type of artificial material, have received much attention recently. Indefinite media are anisotropic unconventionally, with permittivity or permeability tensors that are indefinite—not all of the principal components have the same sign. For such media, the optical dispersion relation, i.e., the equifrequency contour, has the form of a hyperbola (or a hyperboloid in three-dimensional case),[1] significantly differing from those for left-handed media (LHM) and conventional media, which have the form of a circle (or a sphere) or an ellipse (or an ellipsoid). Indefinite media have been shown to possess a variety of unique effects[1–8] and most studies on them are carried out theoretically, based on a homogeneous medium model. However, homogeneous indefinite medium with hyperbolic dispersion evidently leads to some confusing results in physics. For example, in a lossless indefinite medium, propagating waves with a finite frequency may take arbitrarily large wave vector, whereas, for a lossy indefinite medium, the attenuation rate of propagating waves would be inversely proportional to the loss parameter of the medium, if their propagation directions are almost parallel to one of the asymptotes of the dispersion hyperbola. Besides, there exists a special boundary of indefinite medium, whose refraction (or reflection) problem cannot be solved physically, even if the medium is assumed to be lossy.[9] Thus, the characteristic of the dispersion of real indefinite medium materials (IDMMs) still needs to be studied carefully.

Belov *et al.*[10] have discussed the homogenization for a three-dimensional lattice of (magnetic) uniaxial resonant dipoles, but their discussion is limited to the condition of $k_0 a=1$ (k_0 is the wave number in free space and a the internal period), which reflects the state of present IDMM samples at microwave. Later, under a similar condition (i.e., $k_0 a\sim 1$), Wood and Pendry[11] investigated theoretically the dispersion behavior of a specific IDMM, which consists of alternating metal and dielectric layers. Both studies have shown that the dispersion relation for IDMM evidently deviates from the dispersion hyperbola predicted by its effective indefinite medium, even in the wave vector (**k**) region where $|\mathbf{k}|\sim k_0$; and what is more, when the magnitude of negative effective permeability or permittivity is small, an additional propagating mode appears in the anticutoff region predicted by the effective indefinite medium, which cannot exist in any homogeneous indefinite medium. Generally, an artificial material formed by a lattice of artificial "molecules" or inclusions in a matrix can be treated as a homogeneous medium with effective permittivity and permeability under the condition of $a\ll\lambda$, where $\lambda=2\pi/k_0$ being the operating wavelength. In this situation, it is said that the material can be homogenized. For IDMM, however, it seems doubtful that the material can be homogenized with $a\ll\lambda$, as the magnitude of wave vectors for a given frequency may be arbitrarily large in the corresponding indefinite medium, which contradicts the fact that the spatial dispersion effect of IDMM is not again negligible for large wave vectors $|\mathbf{k}|\sim 1/a$.[12] Then, a more practical problem comes: whether the hyperbolic dispersion behavior on the normal-**k** region, where $|\mathbf{k}|\sim k_0$ or $|\mathbf{k}|\ll k_0$, could be realized by IDMM when $a\ll\lambda$, or whether IDMM could be homogenized for the normal-**k** region in this case. So it is necessary to study the effect of internal period on the dispersion of IDMM under the condition of $a\ll\lambda$. In addition, it is also interesting to investigate the refraction problem on the respective boundary of IDMM with $a\ll\lambda$, which corresponds to the special boundary of its effective indefinite medium whose refraction problem cannot be solved physically.

In this paper, the effect of internal period on the dispersion of IDMM is studied under the condition of $a\ll\lambda$, based on a properly simplified physical model for the material, which is formed by a one-dimensional lattice of electric uniaxial dipoles. As the periodicity is retained only in one direction, this material model is easily used to demonstrate the period effect on the dispersion. For such an IDMM, a relatively simple expression is obtained for the dispersion relation, thus, its dispersion behaviors can be investigated analytically. The refraction problem on the special IDMM boundary is also discussed and it is shown to be solvable physically. Finally, the dispersion behaviors predicted by the dipole model are examined numerically by using an example of real IDMM, which consists of alternating metal and dielectric layers.

II. DISPERSION BEHAVIORS OF REAL INDEFINITE MEDIUM MATERIAL

To study the effect of internal period on the optical dispersion of IDMM, we consider a physical model for IDMM as follows: A one-dimensional (1D) periodic array of sheets of electric uniaxial dipoles directed in the x direction in a matrix, as illustrated in the inset of Fig. 1(c). Here, the matrix is a normal medium, and for generality, it is assumed to be anisotropic. In this dipole model, the dipole spacing in the sheets is much smaller than the sheet spacing (a) in the z direction, so the structure can be treated as a system uniform in the xy plane and periodic along z. As the periodicity is retained only in one direction, this material model is easily used to demonstrate the period effect on the dispersion. We consider TE-polarized waves propagating in the xz plane in this IDMM (the results for TM-polarized waves in a similar magnetic dipole structure can be obtained directly from duality). These waves have electric field with one component in the x direction and the other in the z direction, and only the x component interacts with the uniaxial dipoles of the material. The operating wavelength (λ) of the waves is assumed to be much larger than the internal period a.

The electric field of TE waves propagating in the IDMM can be written as $\mathbf{E}=[\hat{x}e_x(z)+\hat{z}e_z(z)]\exp(ik_xx)=[\hat{x}U(z)+\hat{z}V(z)]\exp[i(k_xx+k_zz)]$, where U and V are periodic functions of z (with the period a). Here, the z component (k_z) of the Bloch wave vector ($\mathbf{k}$) is limited to the first Brillouin zone, i.e., $|k_z|\leq\pi/a$, while the x component (k_x) may be arbitrary value, due to the IDMM uniform in the xy plane. By solving Maxwell's equations in the unit cell of the material, we will find the dispersion relation associating the wave vector $\mathbf{k}$ with wave frequency f. The induced polarization in the unit cell can be expressed as $\mathbf{P}=\hat{x}\varepsilon_0 a\alpha E_x\delta(z)$, where the parameter α is a measure for the polarizability of the dipoles. From Maxwell's equations, we obtain

$$\frac{d^2e_x}{dz^2}+p_z^2e_x=-\frac{a\alpha}{\varepsilon_{bx}}p_z^2e_x\delta(z), \quad (1)$$

where $p_z^2=\varepsilon_{bx}(k_0^2-k_x^2/\varepsilon_{bz})$ with $k_0=2\pi f/c$, and ε_{bx} and ε_{bz} are the relative permittivities of the matrix in the x and z directions, respectively. Here, both ε_{bx} and ε_{bz} are normal positive numbers. As the right side of Eq. (1) vanishes at $z\neq 0$, the equation can be easily solved in the following regions: region I, where $-a/2\leq z<0$ and region II, where $0<z\leq a/2$. In region I, the solution of Eq. (1) has the form of

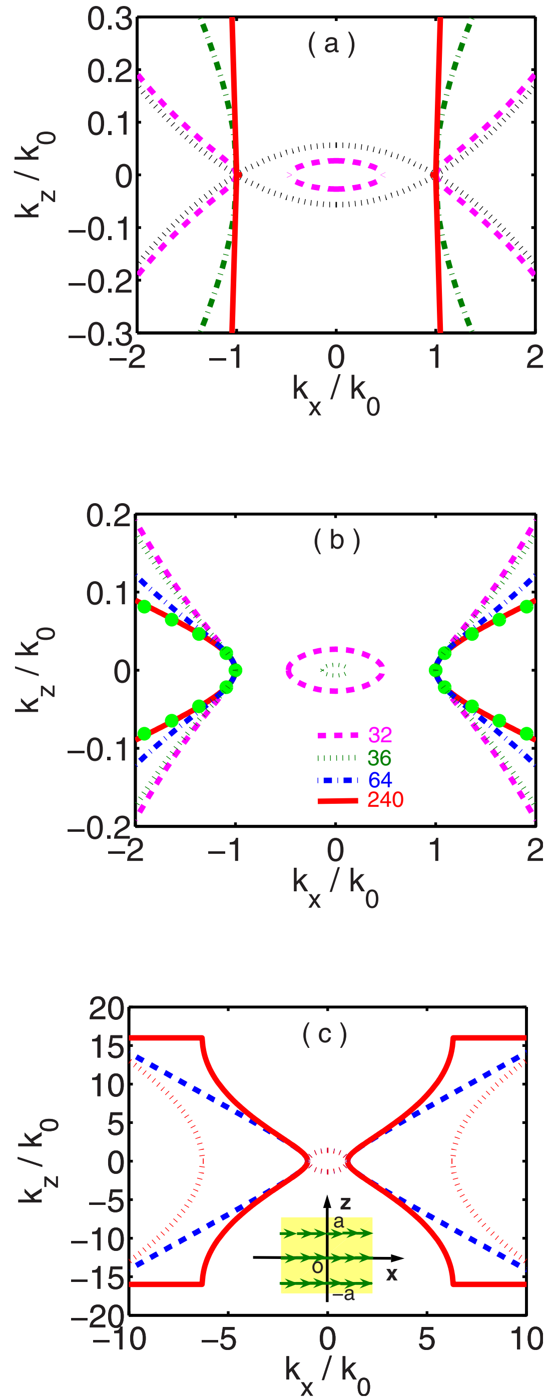

FIG. 1. (Color online) (a) Dispersion curves in the normal-**k** region for the IDMMs with $\varepsilon_\alpha=0$ (dotted), -0.0025 (dashed), -0.1 (dash-dotted), and -1 (solid). The internal period is $a=\lambda/32$ for all cases. (b) Dispersion curves in the normal-**k** region for the IDMMs with different values of λ/a. The parameter $\varepsilon_\alpha=-0.0025$ is kept for all cases. For comparison, the hyperbolic dispersion for the effective indefinite medium is included as solid circles. (c) Dispersion relation for the IDMM with $\varepsilon_\alpha=-1$ and $a=\lambda/32$. Solid and dotted lines represent the real and imaginary parts of k_z, respectively; and dashed lines are the dispersion hyperbola for the effective indefinite medium. Note that $|k_z|=16k_0$ at the border of the Brillouin zone. The inset shows the geometry of IDMM. The common parameters in (a), (b), and (c) are $\varepsilon_{bx}=1$ and $\varepsilon_{bz}=1$.

$e_x^{\mathrm{I}}(z)=A\exp(ip_z z)+B\exp(-ip_z z)$, and obviously, this solution can be extended to the neighboring region of $-a<z\leq -a/2$, in which no dipoles are included. Thus, we express the solution of Eq. (1) for region II as $e_x^{\mathrm{II}}(z)=e_x^{\mathrm{I}}(z-a)\exp(ik_z a)=\{A\exp[ip_z(z-a)]+B\exp[-ip_z(z-a)]\}\exp(ik_z a)$. At the interface between regions I and II, two solutions e_x^{I} and e_x^{II} must satisfy the following matching conditions:

$$e_x^{\mathrm{I}}|_{z\to 0-}=e_x^{\mathrm{II}}|_{z\to 0+}=e_x(0),$$

$$\left.\frac{de_x^{\mathrm{II}}}{dz}\right|_{z\to 0+}-\left.\frac{de_x^{\mathrm{I}}}{dz}\right|_{z\to 0-}=-\frac{a\alpha}{\varepsilon_{bx}}p_z^2 e_x(0).$$

Obviously, the first condition represents the continuity of the x component of the electric field at the interface $z=0$, whereas the second one, which is derived by integrating Eq. (1) over an infinitesimal interval centered at $z=0$, reflects the boundary condition for the y component of the magnetic field ($\mathbf{H}$). From these matching conditions, we have

$$A+B=A\exp[i(k_z-p_z)a]+B\exp[i(k_z+p_z)a], \quad (2)$$

$$i\frac{a\alpha}{\varepsilon_{bx}}p_z(A+B)=\{-1+\exp[i(k_z-p_z)a]\}A+\{1-\exp[i(k_z+p_z)a]\}B. \quad (3)$$

By eliminating the coefficients A and B in Eqs. (2) and (3), we then find the dispersion relation for the IDMM in the form

$$\cos(k_z a)=\cos(p_z a)-\frac{\alpha}{2\varepsilon_{bx}}p_z a\sin(p_z a). \quad (4)$$

The dispersion relation [Eq. (4)] for the IDMM is somewhat complicated. Let's first examine its characteristic in the normal-$\mathbf{k}$ region of interest, where $|\mathbf{k}|\sim k_0$ or $|\mathbf{k}|\ll k_0$. Unlike the previous works,[10,11] we assume that $k_0 a\ll 1$ (i.e., $a\ll\lambda$) throughout this paper. Thus, for the normal-$\mathbf{k}$ region, we have $|\mathbf{k}|a\ll 1$, and Eq. (4) is simplified to

$$k_z^2=\left[\varepsilon_\alpha+\frac{\varepsilon_{bx}^2}{12}\left(1-\frac{2\varepsilon_\alpha}{\varepsilon_{bx}}\right)\left(k_0^2-\frac{k_x^2}{\varepsilon_{bz}}\right)a^2\right]\left(k_0^2-\frac{k_x^2}{\varepsilon_{bz}}\right), \quad (5)$$

where $\varepsilon_\alpha=\alpha+\varepsilon_{bx}$. In the derivation of Eq. (5), higher-order terms are retained in the series expansion for the right side of Eq. (4). This is only because the IDMM is inherently dispersive and its parameter ε_α may become nearly zero at certain frequencies. The quantities $k_z a$ and $(k_x a)^2$ are of the same order in the limiting case of $\varepsilon_\alpha\to 0$. In order to know completely about the dispersion behavior of the IDMM, it is necessary to include such a limiting case in our analysis. Since $k_x^2a^2, k_0^2a^2\ll 1$, and $\varepsilon_{bx},\varepsilon_{bz}\sim 1$, Eq. (5) is written approximately as

$$k_z^2=\left[\varepsilon_\alpha+\frac{\varepsilon_{bx}^2}{12}\left(k_0^2-\frac{k_x^2}{\varepsilon_{bz}}\right)a^2\right]\left(k_0^2-\frac{k_x^2}{\varepsilon_{bz}}\right). \quad (6)$$

If we could neglect the terms involving the factor a in Eq. (6), the dispersion equation further reduces to

$$\frac{k_x^2}{\varepsilon_{bz}}+\frac{k_z^2}{\varepsilon_\alpha}=k_0^2. \quad (7)$$

This is just the dispersion relation for the homogenized IDMM. In this situation, the IDMM is equivalent to a homogeneous medium with the effective permittivities ε_α and ε_{bz} in the x and z directions, respectively. In what follows, we will restrict ourselves to the material case with $\varepsilon_\alpha<0$, i.e., $\alpha<-\varepsilon_{bx}$, so that the dispersion relation for the corresponding homogeneous medium always have the form of a hyperbola, regardless of whether the homogenization of the material is valid or not. Actually, only in this case, it is reasonable to refer to the material as an IDMM. By comparing Eqs. (4), (6), and (7), it is clear that the dispersion relation [Eq. (7)] is only valid for the region of $|\mathbf{k}|\ll 1/a$ and when $|\varepsilon_\alpha|\gg(k_0a)^2$.

It is well known that there exists an anticutoff (k_{ac}) for wave vector of propagating waves in an indefinite medium, i.e., only the waves with $|\mathbf{k}|\geq k_{ac}$ is allowed to propagate. Using the dispersion relation [Eq. (6)], we can analyze the anticutoff problem for the IDMM. The existence of anticutoff means that $k_z^2<0$ at $k_x=0$. Let $k_x=0$, from Eq. (6), we have

$$k_z^2=\left(\varepsilon_\alpha+\frac{\varepsilon_{bx}^2}{12}k_0^2a^2\right)k_0^2. \quad (8)$$

Thus, in the case when $\varepsilon_\alpha<-\varepsilon_{bx}^2k_0^2a^2/12$, the IDMM indeed has an anticutoff; but in the case if $|\varepsilon_\alpha|<\varepsilon_{bx}^2k_0^2a^2/12$, the anticutoff seems not to appear, though ε_α is negative.

We further analyze the dispersion behavior of the IDMM in the normal-$\mathbf{k}$ region. Let $k_z=0$, from Eq. (6), we have

$$\left[\left(1+\frac{12\varepsilon_\alpha}{\varepsilon_{bx}^2k_0^2a^2}\right)-\frac{k_x^2}{\varepsilon_{bz}k_0^2}\right]\left(1-\frac{k_x^2}{\varepsilon_{bz}k_0^2}\right)=0. \quad (9)$$

In the case when $\varepsilon_\alpha<-\varepsilon_{bx}^2k_0^2a^2/12$, where the anticutoff exists, we find the anticutoff $k_{ac}=\sqrt{\varepsilon_{bz}}k_0$, which is the same as that for the corresponding indefinite medium. Note that in this case, perhaps the material cannot be homogenized actually, as the homogenization strictly requires that $\varepsilon_\alpha\ll-\varepsilon_{bx}^2k_0^2a^2/12$. In the case if $|\varepsilon_\alpha|<\varepsilon_{bx}^2k_0^2a^2/12$ (but $\varepsilon_\alpha<0$), where the anticutoff seems not to appear, the situation becomes quite complicated. There are two solutions to Eq. (9), i.e., $k_{x1}=\sqrt{\varepsilon_{bz}(1+12\varepsilon_\alpha/\varepsilon_{bx}^2k_0^2a^2)}k_0$ and $k_{x2}=\sqrt{\varepsilon_{bz}}k_0$. From Eq. (6), we find that $k_z^2>0$ for $|k_x|<k_{x1}$ or $|k_x|>k_{x2}$, while $k_z^2<0$ for $k_{x1}<|k_x|<k_{x2}$. Therefore, in this case, there still exists an anticutoff of $k_{ac}=\sqrt{\varepsilon_{bz}}k_0$, at which $k_z=0$, and waves with $|k_x|$ a little smaller than k_{ac} are forbidden, while waves with $|k_x|\geq k_{ac}$ are propagating. Moreover, an additional propagating mode occurs in the interval of $|k_x|\leq k_{x1}$. Such an additional mode cannot exist in any (homogeneous) indefinite medium. Moreover, for the IDMM, what is more, we find that in the limiting case of $\varepsilon_\alpha\to 0$, the dispersion relation [Eq. (6)] becomes

$$k_z = \pm \frac{\varepsilon_{bx}}{2\sqrt{3}}\left(k_0^2 - \frac{k_x^2}{\varepsilon_{bz}}\right)a, \tag{10}$$

which is a pair of parabolas. Therefore, in this limiting case, the dispersion curve for the IDMM does not tend to be two separated lines as predicted by the effective indefinite medium. Interestingly, no anticutoff exists in this special situation. It should be indicated that the parabolalike dispersion behavior was found in our previous study on the metal-dielectric photonic crystal (with $a=\lambda/3$).[13] To verify our analysis above, using Eq. (4), we numerically calculate the dispersion relation in the normal-**k** region for the IDMMs with various ε_α, and the results are plotted in Fig. 1(a).

From the above analysis, it is clear that for any negative ε_α, the additional propagating mode would not appear when $a<a_c$, where $a_c=(\sqrt{-3\varepsilon_\alpha/\pi\varepsilon_{bx}})\lambda$. In the case of no additional mode, however, the dispersion relation of the IDMM with $a\sim a_c$ still deviates evidently from the dispersion hyperbola for the corresponding effective medium, as seen from Eq. (6). Moreover, only when $a\ll a_c$, the dispersion curve of the IDMM would agree well with the corresponding dispersion hyperbola. These phenomena are illustrated in Fig. 1(b), which shows the dispersion curves for the IDMMs with $\lambda/a=32$, 36, 64, and 240. The matrix is assumed to be air, and the parameter $\varepsilon_\alpha=-0.0025$, for which we find $a_c=\lambda/36.3$. Note that the effective indefinite media are the same for all cases since ε_α is kept constant. For comparison, the dispersion hyperbola for the effective indefinite medium is included as solid circles in Fig. 1(b). As seen from Fig. 1(b), for $a=\lambda/32$, there exists an additional mode in the range of $|k_x|\le 0.47k_0$, and the range of the additional mode becomes very narrow when $a=\lambda/36$. The additional mode already vanishes when $a=\lambda/64$, and in this case, the dispersion curve for the IDMM is even different from the corresponding dispersion hyperbola. Only when the internal period decreases to $a=\lambda/240$, good agreement between the dispersion curves for the IDMM and the corresponding indefinite medium is achieved in the normal-**k** region.

Next, we analyze the dispersion characteristic of the IDMM in the large-**k** region, where $|\mathbf{k}|a\sim 1$ or $|\mathbf{k}|a\gg 1$. As propagating waves for a given frequency can take arbitrarily large wave vector in the effective indefinite medium, it is perhaps possible for waves to propagate with large wave vector in the IDMM. For waves with large wave vectors, the hyperbolic dispersion relation [Eq. (7)] is no longer valid even when $|\varepsilon_\alpha|\gg(k_0a)^2$. It is interesting whether there exists any upper cutoff for wave vector of propagating waves in the IDMM. If, as for the effective indefinite medium, no upper cutoff exists for the IDMM, the dispersion diagram of the material would have a multiband structure; due to that, k_z is limited to the region $[-\pi/a,\pi/a]$. If it is really so, this should be reflected in Eq. (4). For $|k_x|\ge k_{ac}=\sqrt{\varepsilon_{bz}}k_0$, we rewrite Eq. (4) as

$$\cos(k_z a) = g(t)\left[\frac{1}{f(t)} + \frac{\gamma}{2}\right], \tag{11}$$

where $t=\sqrt{-p_z^2}a=\sqrt{\varepsilon_{bx}(k_x^2/\varepsilon_{bz}-k_0^2)}a>0$, $\gamma=\alpha/\varepsilon_{bx}$, and the functions f and g are defined as $f(t)=t\tanh(t)$ and $g(t)=t\sinh(t)$. Note that the parameter $\gamma<-1$ since $\varepsilon_\alpha<0$ or $\alpha<-\varepsilon_{bx}$. The functions f and g have the following properties: At $t=0$, $f=g=0$, but $f^{-1}g=1$; for $t>0$, f and g increase monotonously with t; as $t\to\infty$, $f,g\to\infty$. Thus, at $t=0$ (or $|k_x|=k_{ac}$), $\cos(k_za)=1$, i.e., $k_z=0$; and for $0<t\ll 1$, $\cos(k_za)\approx 1+(1+\gamma)t^2/2$, indicating that $\cos(k_za)$ decreases as t (or $|k_x|$) begins to increase from 0 (or k_{ac}). Obviously, there exists a point t_0, at which $1/f+\gamma/2=0$, i.e., $\cos(k_za)=0$. Moreover, as t increases from t_0, the function $\cos(k_za)<0$, as well as it decreases monotonously with t. Since $\cos(k_za)\approx\gamma g(t)/2\to-\infty$ as $t\to\infty$, there must exist a critical point t_c, at which $\cos(k_za)=-1$, i.e., $k_z=\pm\pi/a$. If t further increases from t_c, then $\cos(k_za)<-1$, meaning that k_z has an imaginary part. Therefore, for k_x, there also exists a critical value

$$k_{uc} = \sqrt{\varepsilon_{bz}\left(k_0^2 + \frac{t_c^2}{a^2\varepsilon_{bx}}\right)}, \tag{12}$$

where t_c only depends on the value of γ, and waves with $|k_x|>k_{uc}$ are evanescent in the IDMM. Clearly, there exists only a single value for t_c (or k_{uc}), at which $\cos(k_za)=-1$, because $\cos(k_za)$ decreases monotonously in the region of $t\ge t_0$, where $\cos(k_za)\le 0$. Therefore, for the IDMM with finite period, the dispersion diagram is not a multiband structure, and there exists physically an upper cutoff, above which, i.e., $|\mathbf{k}|>\sqrt{k_{uc}^2+(\pi/a)^2}$, the wave is forbidden in the material. To illustrate this dispersion behavior, the dispersion relation for the IDMM with a typical parameter $\varepsilon_\alpha=-1$ is plotted in Fig. 1(c), where the dispersion hyperbola for the corresponding indefinite medium is included as dashed lines for comparison. As seen from Fig. 1(c), waves are propagating only for $k_0<|k_x|<8k_0$, and at larger wave vectors, the dispersion curve is also strongly distorted and thus deviates from the dispersion hyperbola.

III. WAVE REFRACTION ON THE SPECIAL IDMM BOUNDARY

The dispersion relation [Eq. (4)] can be used to analyze the refraction behavior of waves incident upon the IDMM boundary. If the IDMM boundary is parallel or normal to the x axis, and a plane wave is incident from air upon the material boundary, the refraction problem is only related to the part of the dispersion curve in the normal-**k** region, which may be approximated by a part of hyperbola generally, as seen in Fig. 1(c). In this case, the IDMM can be represented by its effective medium. The wave refraction or reflection on the indefinite medium boundary parallel or normal to the anisotropy axis was well discussed in Ref. 3, and the phenomenon of anomalous total reflection was found. Evidently, such a phenomenon generally also happens to the IDMM boundary parallel to the x axis, and the normally incident wave with $k_x=0$ will be totally reflected, as waves with $|k_x|<k_{ac}$ are forbidden in the IDMM. However, for the IDMM with small $|\varepsilon_\alpha|$ when the additional propagating mode exists, the normally incident wave is not again totally reflected by the IDMM boundary. For this case, the prediction from the effective indefinite medium is wrong completely. Interestingly, in this case, only these waves with incident angle

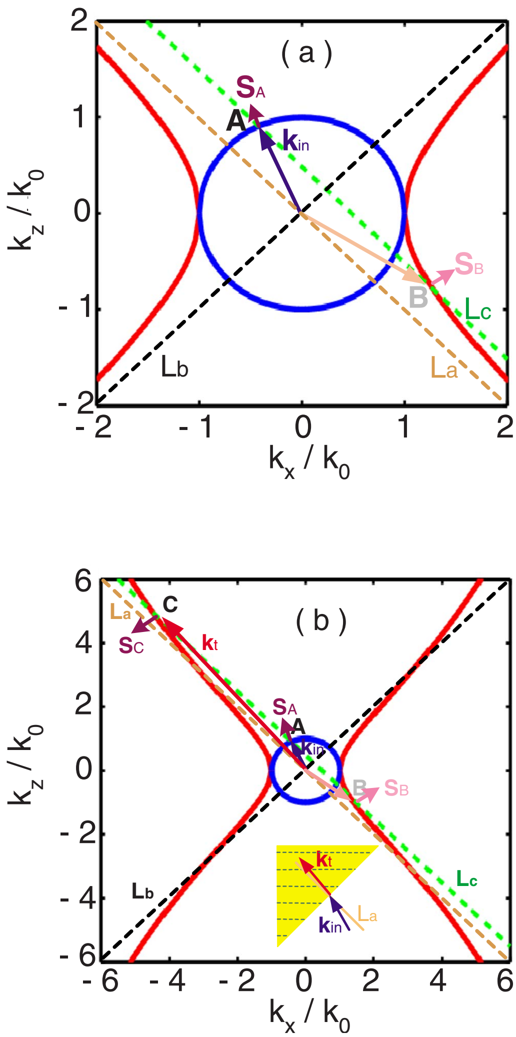

FIG. 2. (Color online) Wave vector diagram for wave refraction on the special boundary of (a) the effective indefinite medium and (b) the IDMM with $a=\lambda/32$. The circle is the dispersion curve for air and $\mathbf{k}_{\text{in}}$ is the incident wave vector. L_a is an asymptote of the dispersion hyperbola for the indefinite medium and L_b, which is orthogonal to L_a, indicates the orientation of the medium or material boundary. L_c is the construction line, representing the conservation of the wave vector component parallel to L_b. A and B in both (a) and (b), and C in (b), are the intersections between L_c and the dispersion curves, and $\mathbf{S}_A$, $\mathbf{S}_B$, and $\mathbf{S}_C$ are the Poynting vectors of these points, respectively. The wave vector ($\mathbf{k}_t$) at C corresponds to a causal refracted wave in the IDMM. The parameters of the IDMM are the same as in Fig. 1(c).

larger than one critical angle but smaller than another may be totally reflected by the IDMM boundary.

It is of particular interest to discuss the wave refraction on the special IDMM boundary, which is perpendicular to one of the asymptotes of the dispersion hyperbola for the effective indefinite medium. Let's first examine the refraction problem for the case of the indefinite medium. The wave vector diagram for analyzing the wave refraction is shown in Fig. 2(a), where a plane wave with wave vector $\mathbf{k}_{\text{in}}$ is incident from air upon the special boundary of the indefinite medium [see the inset of Fig. 2(b)]. The orientation of the boundary is indicated by the dashed line L_b, which is orthogonal to the asymptote (L_a) of the dispersion hyperbola. The construction line L_c represents the conservation of the wave vector component parallel to L_b, which must be satisfied for wave transmission across the interface. As seen from Fig. 2(a), L_c only intersects one point (B) of the dispersion hyperbola, and its Poynting vector ($\mathbf{S}_B$) points toward the source, indicating that it contributes to no refracting wave. Thus, a puzzling problem arises about the refraction on this special boundary. Next, we investigate the case of the IDMM under consideration, and the wave vector diagram is plotted in Fig. 2(b). In contrast to the corresponding dispersion hyperbola, the dispersion curve for the IDMM is distorted at large wave vectors, which makes the construction line L_c intersect two points B and C, with Poynting vectors $\mathbf{S}_B$ and $\mathbf{S}_C$, respectively. $\mathbf{S}_B$ points toward the source as in the homogeneous case, but $\mathbf{S}_C$ points away from the source. Thus, point C contributes to a refracting wave. So the puzzling problem mentioned above vanishes for the IDMM with finite internal period.

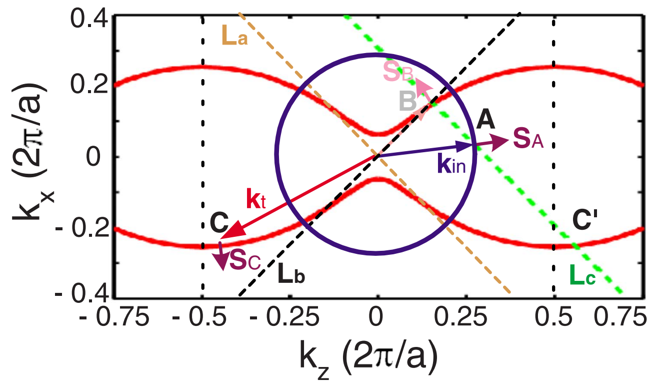

FIG. 3. (Color online) Wave vector diagram for wave refraction on the special IDMM boundary. The circle is the dispersion curve for the incident medium with a high index, and $\mathbf{k}_{\text{in}}$ is the incident wave vector. L_a is an asymptote of the dispersion hyperbola for the effective indefinite medium, and L_b, which is orthogonal to L_a, indicates the orientation of the material boundary. L_c is the construction line, and it intersects point B in the first Brillouin zone and point C' in a high-order Brillouin zone. C' falls onto C when folded back to the first zone. $\mathbf{S}_A$, $\mathbf{S}_B$, and $\mathbf{S}_C$ are the Poynting vectors of A, B, and C, respectively. The wave vector ($\mathbf{k}_t$) at C corresponds to a causal refracted wave in the IDMM. The parameters of the IDMM are the same as in Fig. 1(c).

However, as the dispersion curve for the IDMM is bounded and not closed, it still seems possible that no refracted wave might be case for the special IDMM boundary if the plane wave is incident from a medium with very high index and has a very large wave vector component parallel to the boundary, as illustrated in Fig. 3. In this case, the construction line L_c intersects only one point (B) of the dispersion curve for the IDMM, and it, with Poynting vector $\mathbf{S}_B$ pointing toward the source, contributes to no refracting wave. In this situation, we need to repeat the dispersion curve in higher-order Brillouin zones, as refracted wave in the IDMM is a Bloch wave.[14,15] As seen from Fig. 3, L_c can always intersect a point (C') in a high-order zone. To acquire the fundamental wave vector of the corresponding Bloch

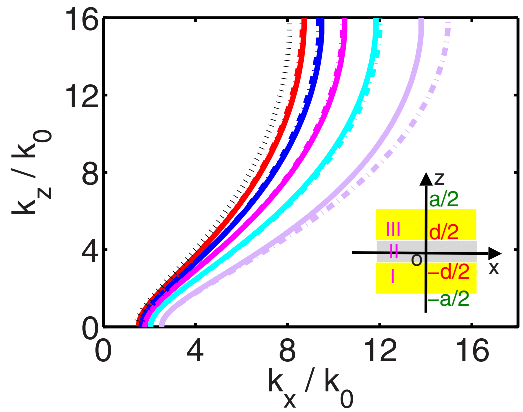

FIG. 4. (Color online) Dispersion curves (solid lines) for the layered IDMMs with various f_m. The parameter $\varepsilon_t=-2$ is kept constant for all cases. Dotted line is the dispersion curve for the corresponding dipole model. Dot-dashed lines are the dispersion curves for the effective dipole models for different f_m. For the solid or dot-dashed lines from left to right: f_m=0.1, 0.2, 0.3, 0.4, and 0.5, respectively. The other parameters are ε_r=2.1 and $a=\lambda/32$. The inset shows the unit cell of layered IDMM.

wave, the point C' is folded back to the first zone by subtracting a reciprocal lattice vector, and it falls onto point C. Point C has a Poynting vector $\mathbf{S}_C$ pointing away from the source, thus contributes to a refracting wave. Therefore, for the IDMM with finite internal period, there really exists no special boundary, on which the refraction problem cannot be solved physically.

IV. LAYERED IDMM EXAMPLE

All analysis performed above is based on the 1D dipole model for IDMM. Now, we consider a real example of IDMM to examine the predicted dispersion behaviors, and also to compare it with its dipole model. The real IDMM consists of alternating layers of metal with relative permittivity $\varepsilon_m<0$ and dielectric with relative permittivity $\varepsilon_r>0$. The inset of Fig. 4 shows the unit cell of the material. The internal period (a) of the material is assumed to be much smaller than the operating wavelength. According to the effective medium theory,[16] this material can be designed as an indefinite medium with negative effective permittivity ε_t in the xy plane and positive effective permittivity ε_n in the z direction, which are given by $\varepsilon_t=f_m\varepsilon_m+(1-f_m)\varepsilon_r$ and $\varepsilon_n=\varepsilon_r\varepsilon_m/[f_m\varepsilon_r+(1-f_m)\varepsilon_m]$, where $f_m=d/a$ being the filling fraction of the metal, and d is the metal layer width. For TE-polarized waves propagating in the xz plane, the dispersion relation for this indefinite medium then has a hyperbolic form. Let us strictly solve for the dispersion relation for this layered IDMM. For this purpose, we first divide the unit cell of the material into three regions: region I, where $-a/2\le z<-d/2$; region II, where $-d/2\le z\le d/2$; and region III where $d/2<z\le a/2$. Then, the magnetic field component (H_y) of (Bloch) waves with wave vector $\mathbf{k}=\hat{x}k_x+\hat{z}k_z$ in three regions can be written as

$$H_y^{\mathrm{I}}(x,z)=[A_1\exp(ip_zz)+B_1\exp(-ip_zz)]\exp(ik_xx),$$

$$H_y^{\mathrm{II}}(x,z)=[A_2\exp(-q_zz)+B_2\exp(q_zz)]\exp(ik_xx),$$

$$H_y^{\mathrm{III}}(x,z)=H_y^{\mathrm{I}}(x,z-a)\exp(ik_za),$$

where $p_z=\sqrt{\varepsilon_rk_0^2-k_x^2}$ and $q_z=\sqrt{k_x^2-\varepsilon_mk_0^2}$. The nonzero components (E_x and E_z) of the electric field can be obtained straightforwardly from H_y. The dispersion relation for the layered IDMM is determined by imposing matching conditions on the parallel field components (H_y and E_x) at the interface between regions I and II and the one between regions II and III. From the continuity of H_y and E_x at $z=-d/2$, we have

$$\bar{A}_1+\bar{B}_1=\bar{A}_2+\bar{B}_2, \tag{13}$$

$$-i\kappa(\bar{A}_1-\bar{B}_1)=\bar{A}_2-\bar{B}_2, \tag{14}$$

where $\bar{A}_1=A_1\exp(-ip_zd/2)$, $\bar{B}_1=B_1\exp(ip_zd/2)$, $\bar{A}_2=A_2\exp(q_zd/2)$, $\bar{B}_2=B_2\exp(-q_zd/2)$, and $\kappa=\varepsilon_mp_z/\varepsilon_rq_z$. Then, we obtain

$$\bar{A}_2=\frac{1}{2}(1-i\kappa)\bar{A}_1+\frac{1}{2}(1+i\kappa)\bar{B}_1, \tag{15}$$

$$\bar{B}_2=\frac{1}{2}(1+i\kappa)\bar{A}_1+\frac{1}{2}(1-i\kappa)\bar{B}_1. \tag{16}$$

Similarly, from the matching conditions at $z=d/2$, we can obtain

$$\bar{A}_2\exp(-q_zd)=\frac{1}{2}\{(1-i\kappa)\exp[-ip_z(a-d)]\bar{A}_1+(1+i\kappa)\exp[ip_z(a-d)]\bar{B}_1\}\exp(ik_za), \tag{17}$$

$$\bar{B}_2\exp(q_zd)=\frac{1}{2}\{(1+i\kappa)\exp[-ip_z(a-d)]\bar{A}_1+(1-i\kappa)\exp[ip_z(a-d)]\bar{B}_1\}\exp(ik_za). \tag{18}$$

Substitution of Eqs. (15) and (16) into Eqs. (17) and (18) yields

$$(1-i\kappa)\{\exp(-q_zd)-\exp[-ip_z(a-d)]\exp(ik_za)\}\bar{A}_1=-(1+i\kappa)\{\exp(-q_zd)-\exp[ip_z(a-d)]\exp(ik_za)\}\bar{B}_1, \tag{19}$$

$$(1+i\kappa)\{\exp(q_zd)-\exp[-ip_z(a-d)]\exp(ik_za)\}\bar{A}_1=-(1-i\kappa)\{\exp(q_zd)-\exp[ip_z(a-d)]\exp(ik_za)\}\bar{B}_1. \tag{20}$$

By eliminating the coefficients $\bar{A}_1$ and $\bar{B}_1$ in Eqs. (19) and (20), we can finally obtain the dispersion relation for the layered IDMM in the form

$$\cos(k_z a) = \cos[(1-f_m)p_z a]\cosh(f_m q_z a) - \frac{1}{2}\left(\kappa - \frac{1}{\kappa}\right)\sin[(1-f_m)p_z a]\sinh(f_m q_z a). \tag{21}$$

The dispersion equation [Eq. (21)] for the layered IDMM seems quite different from Eq. (4), the one for the dipole model. However, if we let $f_m \to 0$ and $\varepsilon_m \to -\infty$ but keep ε_t constant, we find that $\varepsilon_n = \varepsilon_r$ in this limiting case, and Eq. (21) reduces to Eq. (4), with $\varepsilon_\alpha = \varepsilon_t$ and $\varepsilon_{bx} = \varepsilon_{bz} = \varepsilon_r$. Therefore, the layered IDMM can be described exactly by the dipole model in the limiting case.

It is interesting to investigate how the metal layer width affects the dispersion of the layered IDMM. For this purpose, we numerically calculate the dispersion relation of the layered IDMM using Eq. (21). As an example, the dielectric layers are assumed to be fused silica (SiO_2) with $\varepsilon_r = 2.1$, and the internal period is $a = \lambda/32$. We fix $\varepsilon_t = -2$, but vary f_m to calculate the dispersion relation for each case. The results are plotted as solid lines in Fig. 4, where the dispersion curve for the corresponding dipole model, with $\varepsilon_\alpha = \varepsilon_t$ and $\varepsilon_{bx} = \varepsilon_{bz} = \varepsilon_r$, is also included as dotted line for comparison. As the dispersion diagram is symmetric about both the k_x and k_z axes, only its quarter is shown in Fig. 4 for clarity. As seen from Fig. 4, when f_m increases, the dispersion curve for the layered IDMM moves toward the large k_x region and differs from the dispersion curve for the dipole model even in the normal-**k** region. However, if we introduce an effective dipole model for each case, with $\varepsilon_\alpha = \varepsilon_t$, $\varepsilon_{bx} = \varepsilon_r - (\varepsilon_r - 1)f_m$, and $\varepsilon_{bz} = \varepsilon_n$, then we find that in the normal-**k** region, the dispersion curve (dot-dashed line) for the effective dipole model is always in good agreement with that for the layered IDMM, as expected. Moreover, for $f_m \leq 0.4$, two dispersion curves are almost the same in the whole **k** region. This means that the finite width effect of the metal layers is almost equivalent to a modification to the matrix for the effective (uniaxial) dipole model of the IDMM. Evidently, the complicated dispersion relation [Eq. (21)] can be well approximated by Eq. (4) with parameters chosen properly, when the metal layers are thin compared to the period.

Unlike the dipole model, the layered IDMM generally has no simple expression for its dispersion relation in the normal-**k** region where $|\mathbf{k}|a \ll 1$. For the normal-**k** region, if we expand both sides of Eq. (21) in Taylor series to second order, then we obtain

$$1 - \frac{1}{2}(k_z a)^2 = 1 - \frac{\varepsilon_t}{2}(k_0 a)^2 + \frac{\varepsilon_t}{2\varepsilon_n}(k_x a)^2. \tag{22}$$

Consequently, we have

$$\frac{k_x^2}{\varepsilon_n} + \frac{k_z^2}{\varepsilon_t} = k_0^2, \tag{23}$$

which agrees with the dispersion relation for the corresponding effective medium. Evidently, Eq. (22) is valid only under the condition of $|\varepsilon_t| \gg (k_0 a)^2$, and so is Eq. (23). When this condition is not satisfied, the right side of Eq. (21) should be expanded to fourth order for the normal-**k** region, and the resulting equation is quite complicated. However, in the limiting case of $\varepsilon_t \to 0$, the resulting equation has a simple form of

$$k_z = \pm \frac{(1-f_m)\varepsilon_r}{2\sqrt{3}}\left(k_0^2 - \frac{k_x^2}{\varepsilon_n}\right)a, \tag{24}$$

which corresponds to the parabolic dispersion. The comparison between Eqs. (24) and (10) implies that the effective dipole model for layered IDMM should take an effective parameter $\varepsilon_{bx} = (1-f_m)\varepsilon_r$ for small $|\varepsilon_t|$, instead of $\varepsilon_{bx} = (1-f_m)\varepsilon_r + f_m$.

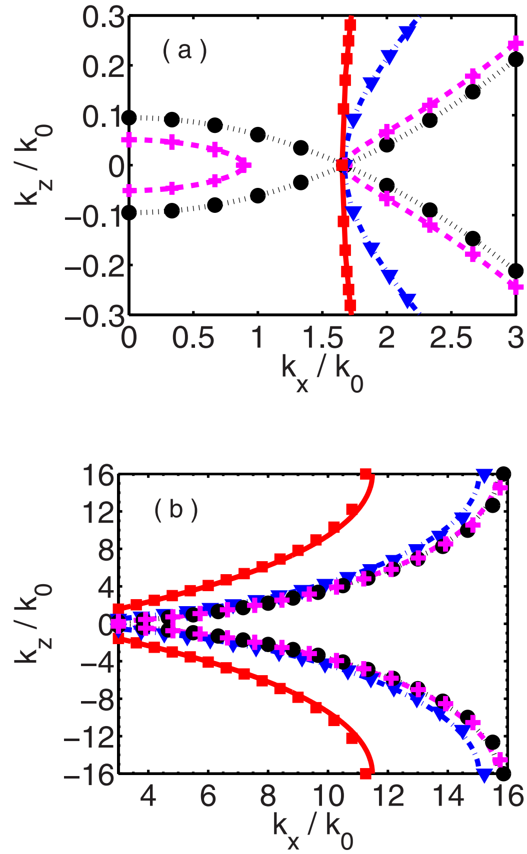

FIG. 5. (Color online) Dispersion curves for the layered IDMMs with $\varepsilon_t = 0$ (dotted), -0.005 (dashed), -0.1 (dash-dotted), and -1 (solid). For comparison, the results for the effective dipole models are included as marks, and good agreement is achieved for each case. In (b), the dispersion curves for $\varepsilon_t = 0$ and -0.005 is not distinguishable from each other. The other parameters of the layered IDMM are $f_m = 0.2$, $\varepsilon_r = 2.1$, and $a = \lambda/32$.

Using Eq. (21), we analyze numerically the dispersion characteristic of the layered IDMM in detail, and find that all dispersion phenomena predicted previously for the dipole model, such as anticutoff, upper cutoff, and additional propagating mode, may also happen to the layered IDMM. To show this, the dispersion curves for the layered IDMMs with $\varepsilon_t = 0$, -0.0065, -0.1, and -1 are plotted in Fig. 5, where we set $f_m = 0.2$ for all cases. For comparison, the results for the effective dipole models are also included as marks in Fig. 5. As the dispersion diagram is symmetric about the k_z axis, only its right half is shown. The effective dipole model has

$\varepsilon_{\alpha}=\varepsilon_t$ and $\varepsilon_{bz}=\varepsilon_n$ for all cases, but takes $\varepsilon_{bx}=(1-f_m)\varepsilon_r$ for $\varepsilon_t=0$, -0.0065, and -0.1, and $\varepsilon_{bx}=(1-f_m)\varepsilon_r+f_m$ for $\varepsilon_t=-1$. As seen from Figs. 5(a) and 5(b), excellent agreement between the layered IDMM and its effective dipole model is achieved for each case, indicating that the layered IDMM, which is perhaps the simplest structure of real IDMM, can be effectively described by our dipole model.

V. SUMMARY

Based on a simplified dipole model for the material, the effect of internal period on the optical dispersion of IDMM has been studied analytically, under the condition of the period much smaller than the operating wavelength. It has been shown that under this condition, the dispersion behavior of IDMM is still deeply affected by its internal period. Interesting phenomena associated with the internal period, such as additional propagating mode in the anticutoff region predicted by the effective indefinite medium, upper cutoff for wave vector of propagating waves, and wave refraction on the special material boundary, have been illustrated in detail. Unlike the homogeneous indefinite medium, there exists no special boundary for IDMM with finite period, on which the refraction problem cannot be solved physically. Our analysis has also shown that for the normal-**k** region of interest, IDMM can always be homogenized as an effective indefinite medium, as long as its period is enough small compared to the wavelength. All these dispersion properties have been verified by one kind of real IDMM, which consists of alternating metal and dielectric layers. In addition, a new type of optical dispersion, i.e., the parabolic dispersion, which can be realized by the IDMM in the limiting case, has been found for the first time. The special material with parabolic dispersion and its application will be discussed detailedly in our next paper.

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*lfshen@zju.edu.cn

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