

Chaos excited chaos synchronizations of integral and fractional order generalized van der Pol systems

Zheng-Ming Ge ^{*}, Mao-Yuan Hsu

*Department of Mechanical Engineering, National Chiao Tung University, 1001 Ta Hsuej Road,
Hsinchu 30050, Taiwan, ROC*

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Abstract

In this paper, chaos excited chaos synchronizations of generalized van der Pol systems with integral and fractional order are studied. Synchronizations of two identified autonomous generalized van der Pol chaotic systems are obtained by replacing their corresponding exciting terms by the same function of chaotic states of a third nonautonomous or autonomous generalized van der Pol system. Numerical simulations, such as phase portraits, Poincaré maps and state error plots are given. It is found that chaos excited chaos synchronizations exist for the fractional order systems with the total fractional order both less than and more than the number of the states of the integer order generalized van der Pol system. © 2006 Elsevier Ltd. All rights reserved.

1. Introduction

Fractional calculus is an old mathematical topic from 17th century. Although it has a long history, applications are only recently focus of interest. Many systems are known to display fractional order dynamics, such as viscoelastic systems [1], dielectric polarization, electrode–electrolyte polarization, and electromagnetic waves. Furthermore, some systems had been found with chaotic motions in the fractional orders. There is a new topic to investigate the control and dynamics of fractional order dynamical systems recently. The behavior of nonlinear chaotic systems when their models become fractional was also investigated widely and reported [2–6].

Sensitive dependence on initial conditions is an important exhibit characteristic of chaotic systems. For this reason, chaotic systems are difficult to be synchronized or controlled. Research in the area of the synchronization of dynamical systems has been widely explored in a variety of fields including physical, chemical and ecological systems, secure communications and so on. There are many control methods to synchronize chaotic systems such as adaptive control, sliding mode control, observer-based design methods, impulsive control and other control methods. There are more advantages in synchronization of uncoupled chaotic systems than that of coupled chaotic systems. And a new uncoupled method of synchronization is presented in this paper. Chaos excited chaos synchronizations of generalized van der Pol systems with

^{*} Corresponding author. Tel.: +886 3 5712121; fax: +886 3 5720634.
E-mail address: zmg@cc.nctu.edu.tw (Z.-M. Ge).

integral and fractional order are studied. Synchronizations of two identified autonomous generalized van der Pol chaotic systems are obtained by replacing their corresponding exciting terms by the same function of chaotic states of a third non-autonomous or autonomous system. Numerical simulations, such as phase portraits, Poincaré maps and state error plots are given. It is found that chaos excited chaos synchronizations exist for the fractional order systems with the total fractional order both less than and more than the number of the states of the integer order generalized van der Pol system.

This paper is organized as follows. In Section 2, the review and the approximation of the fractional order operator is presented. In Section 3, the schemes of chaos excited chaos synchronization of two generalized van der Pol systems by replacing their corresponding terms by the same function of chaotic states of a third nonautonomous or autonomous generalized van der Pol system are given. In Section 4, numerical simulations, such as phase portraits, Poincaré maps error states plots, of synchronizations of various fractional order generalized van der Pol systems are presented. In Section 5, conclusions are drawn.

2. The review and the approximation of fractional-order operators

There are many ways to define a fractional differential operator [7–9]. The commonly used definition for general fractional derivative is the Riemann–Liouville definition. The Riemann–Liouville definition of the fractional-order derivative is

$$D^\alpha y(t) = \frac{d^n}{dt^n} D^{n-\alpha} y(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t \frac{y(\tau)}{(t-\tau)^{\alpha-n+1}} d\tau \quad (1)$$

where $\Gamma(\cdot)$ is a gamma function and n is an integer such that $n-1 \leq \alpha < n$. This definition is different from the usual intuitive definition of derivative.

Thus, it is necessary to develop approximations to the fractional operators using the standard integer order operators. Fortunately, the Laplace transform which is basic engineering tool for analyzing linear systems is still applicable and works:

$$L\left\{\frac{d^\alpha f(t)}{dt^\alpha}\right\} = s^\alpha L\{f(t)\} - \sum_{k=0}^{n-1} s^k \left[\frac{d^{n-1-k} f(t)}{dt^{n-1-k}} \right]_{t=0}, \quad \text{for all } \alpha \quad (2)$$

where n is an integer such that $n-1 \leq \alpha < n$. Upon considering the initial conditions to be zero, this formula reduces to the more expected form

$$L\left\{\frac{d^\alpha f(t)}{dt^\alpha}\right\} = s^\alpha L\{f(t)\} \quad (3)$$

Using the algorithm in [3,10], linear transfer function of approximations of the fractional integrator is adopted. Basically the idea is to approximate the system behavior based on frequency domain arguments. From [11], we get the table of approximating transfer functions for $1/s^\alpha$ with different fractional orders, $\alpha = 0.1\text{--}0.9$, in steps of 0.1, which give the maximum error 2 dB in calculations. These approximations will be used in the following study.

3. Schemes of chaos excited chaos synchronizations of integral and fractional order generalized van der Pol systems

The generalized van der Pol system [12–18] is a nonautonomous system:

$$\begin{aligned} \frac{dx_1}{dt} &= x_2 \\ \frac{dx_2}{dt} &= -x_1 - \varepsilon(1-x_1^2)(c-ax_1^2)x_2 + b \sin \omega t \end{aligned} \quad (4)$$

where ε , a , b , c are parameters, and ω is the circular frequency of the external excitation $b \sin \omega t$. The corresponding nonautonomous fractional order system is

$$\begin{aligned} \frac{d^\alpha x_1}{dt^\alpha} &= x_2 \\ \frac{d^\beta x_2}{dt^\beta} &= -x_1 - \varepsilon(1-x_1^2)(c-ax_1^2)x_2 + b \sin \omega t \end{aligned} \quad (5)$$

where α , β are fractional numbers.

A modified version of Eq. (5) is now proposed. The nonautonomous generalized fractional order van der Pol system (5) with two states is transformed into an autonomous generalized fractional order van der Pol system with three states:

$$\begin{aligned}\frac{d^\alpha x_1}{dt^\alpha} &= x_2 \\ \frac{d^\beta x_2}{dt^\beta} &= -x_1 - \varepsilon(1 - x_1^2)(c - ax_1^2)x_2 + b \sin \omega x_3 \\ \frac{d^\gamma x_3}{dt^\gamma} &= 1\end{aligned}\quad (6)$$

where α, β, γ are fractional numbers, in which the original time t in Eq. (5) is changed to a new state x_3 . When $\gamma = 1$, $x_3 = t$, Eq. (6) reduces to Eq. (5).

Two methods of chaos excited chaos synchronization [19–62] are proposed. In Case 1, two identical generalized fractional order van der Pol systems to be synchronized are

$$\begin{aligned}\frac{d^\alpha x_1}{dt^\alpha} &= x_2 \\ \frac{d^\beta x_2}{dt^\beta} &= -x_1 - \varepsilon(1 - x_1^2)(c - ax_1^2)x_2 + Z\end{aligned}\quad (7)$$

and

$$\begin{aligned}\frac{d^\alpha y_1}{dt^\alpha} &= y_2 \\ \frac{d^\beta y_2}{dt^\beta} &= -y_1 - \varepsilon(1 - y_1^2)(c - ay_1^2)y_2 + Z\end{aligned}\quad (8)$$

where the exciting term $b \sin \omega t$ in Eq. (5) is replaced in Eqs. (7) and (8) by Z which is a function of the chaotic states of a third nonautonomous chaotic system

$$\begin{aligned}\frac{d^\alpha z_1}{dt^\alpha} &= z_2 \\ \frac{d^\beta z_2}{dt^\beta} &= -z_1 - \varepsilon(1 - z_1^2)(c - az_1^2)z_2 + b \sin(\omega t)\end{aligned}\quad (9)$$

Z is chosen as $Z = bgz_1$, $Z = bgz_2$, $Z = gz_1 \sin(\omega t)$, $Z = gz_2 \sin(\omega t)$, respectively, where g is a constant with different values. Error states are defined: $e_1 = x_1 - y_1$, $e_2 = x_2 - y_2$.

In Case 2, two identical generalized fractional order van der Pol systems to be synchronized remain unchanged, as Eqs. (7) and (8). But Z is a function of the chaotic states of a third autonomous system

$$\begin{aligned}\frac{d^\alpha z_1}{dt^\alpha} &= y_2 \\ \frac{d^\beta z_2}{dt^\beta} &= -z_1 - \varepsilon(1 - z_1^2)(c - az_1^2)z_2 + b \sin(\omega z_3) \\ \frac{d^\gamma z_3}{dt^\gamma} &= 1\end{aligned}\quad (10)$$

Z is chosen as $Z = bgz_1$, $Z = bgz_2$, $Z = bg \exp(z_1)$, $Z = bg \exp(z_2)$, $Z = g \exp(z_1) \sin(\omega t)$, $Z = g \exp(z_2) \sin(\omega t)$, respectively. g is a constant with different values.

4. Numerical simulations for the synchronization of fractional order generalized van der Pol systems

The systems to be synchronized are systems (7) and (8) in following two cases. The parameters $a = 3$, $b = 1.0091$, $c = 1.2$ and $d = 0.07$ of system Eqs. (7)–(10) are fixed. Our study of two cases consists of 10 parts:

Case 1: The third system is a nonautonomous system with two states, Eq. (9).

Part (1):

$Z = bgz_1$ where z_1 is the chaotic state of system (9), g is an adjustable constant. When (1) $\alpha = \beta = 1.1$, $\omega = 0.445$, $g = 1.5$; (2) $\alpha = \beta = 1$, $\omega = 0.1301$, $g = 1$; (3) $\alpha = \beta = 0.9$, $\omega = 0.132$, $g = 1$; (4) $\alpha = \beta = 0.8$, $\omega = 0.1315$, $g = 0.8$; (5)

$\alpha = \beta = 0.7$, $\omega = 0.31812$, $g = 0.3$, chaos synchronization is obtained. For saving space only the phase portraits and Poincaré maps of the fractional order synchronized systems and time history of error states plot for condition (5) are shown in Fig. 1. When α, β take the fractional number less than 0.7, no chaotic synchronization is found.

Part (2):

$Z = bgz_2$ where z_2 is the chaotic state of system (9). When (1) $\alpha = \beta = 1.1$, $\omega = 0.445$, $g = 0.8$; (2) $\alpha = \beta = 1$, $\omega = 0.12961875$, $g = 3$; (3) $\alpha = \beta = 0.9$, $\omega = 0.132$, $g = 9$; (4) $\alpha = \beta = 0.8$, $\omega = 0.1315$, $g = 13$; (5) $\alpha = \beta = 0.7$, $\omega = 0.31812$, $g = 14.65$, chaos synchronization is obtained. For saving space only the phase portraits and Poincaré maps of the fractional order synchronized systems and time history of error states plot for condition (5) are shown in Fig. 2. When α, β take the fractional number less than 0.7, no chaotic synchronization is found.

Part (3):

$Z = gz_1 \sin(\omega t)$ where z_1 is the chaotic state of system (9). When (1) $\alpha = \beta = 1.1$, $\omega = 0.445$, $g = 1.5$; (2) $\alpha = \beta = 1$, $\omega = 0.12961875$, $g = 3$; (3) $\alpha = \beta = 0.9$, $\omega = 0.132$, $g = 0.5$; (4) $\alpha = \beta = 0.8$, $\omega = 0.1315$, $g = 5$; (5) $\alpha = \beta = 0.7$, $\omega = 0.31812$, $g = 10$, chaos synchronization is obtained. For saving space only the phase portraits and Poincaré maps of the fractional order synchronized systems and time history of error states plot for condition (5) are shown in Fig. 3. When α, β take the fractional number less than 0.7, no chaotic synchronization is found.

Part (4):

$Z = gz_2 \sin(\omega t)$ where z_2 is the chaotic state of system (9). When (1) $\alpha = \beta = 1.1$, $\omega = 0.445$, $g = 1$; (2) $\alpha = \beta = 1$, $\omega = 0.12961875$, $g = 3$; (3) $\alpha = \beta = 0.9$, $\omega = 0.132$, $g = 1$; (4) $\alpha = \beta = 0.8$, $\omega = 0.1315$, $g = 0.5$; (5) $\alpha = \beta = 0.7$, $\omega = 0.31812$, $g = 0.5$, chaos synchronization is obtained. For saving space only the phase portraits and Poincaré maps of the fractional order synchronized systems and time history of error states plot for condition (5) are shown in Fig. 4. When α, β take the fractional number less than 0.7, no chaotic synchronization is found.

Case 2: The third system is an autonomous system with three states, Eq. (10).

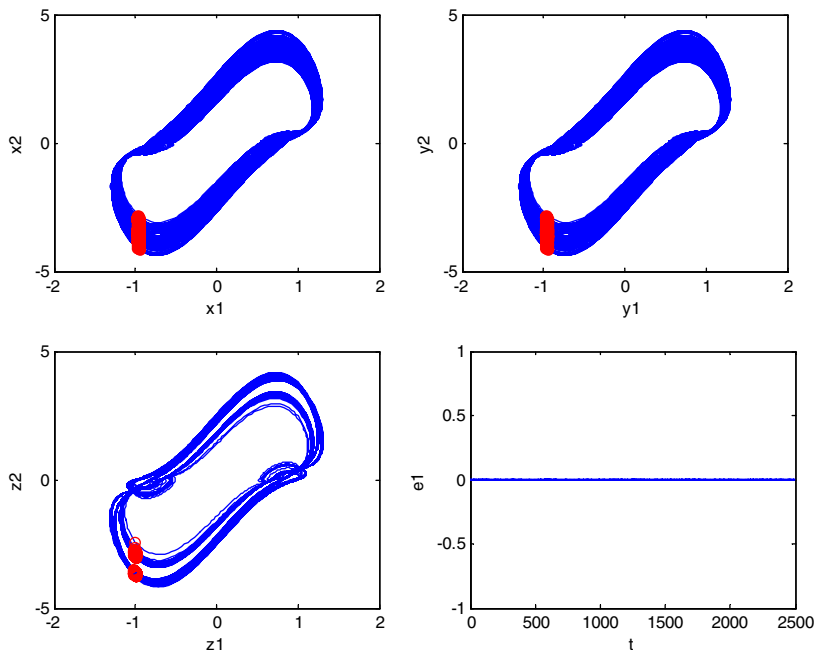


Fig. 1. Phase portraits and Poincaré maps of the synchronized fractional order systems and time history of states error for $Z = bgz_1$ with order $\alpha = \beta = 0.7$, $\omega = 0.31812$, $g = 0.3$.

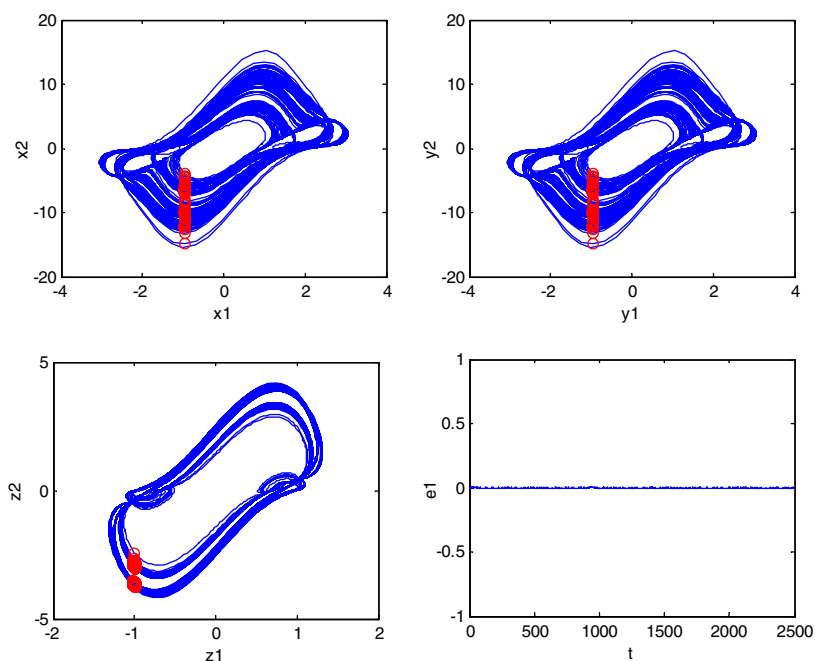


Fig. 2. Phase portraits and Poincaré maps of the synchronized fractional order systems and time history of states error for $Z = bgz_2$ with order $\alpha = \beta = 0.7$, $\omega = 0.31812$, $g = 14.65$.

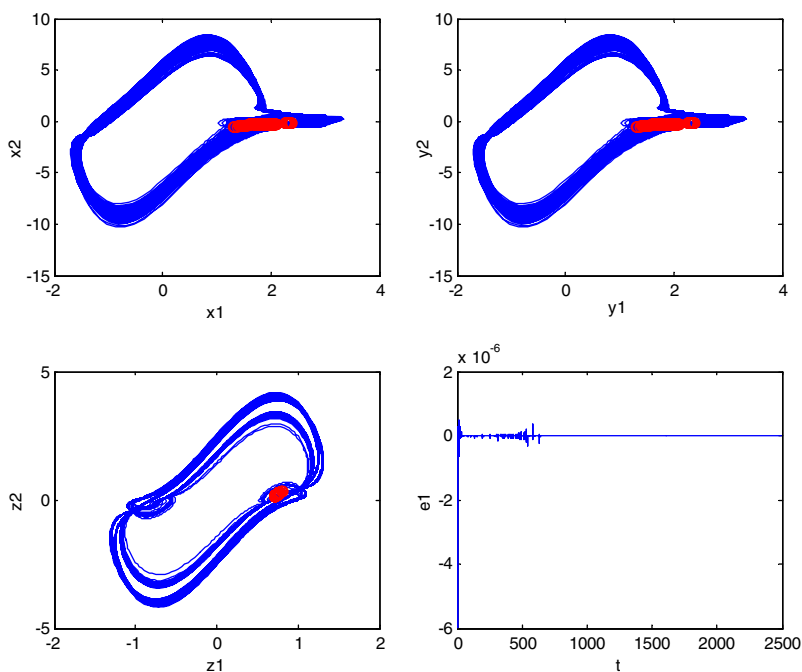


Fig. 3. Phase portraits and Poincaré maps of the synchronized fractional order systems and time history of states error for $Z = gz_1 \sin(\omega t)$ with order $\alpha = \beta = 0.7$, $\omega = 0.31812$, $g = 10$.

Part (1):

$Z = bgz_1$ where z_1 is the chaotic state of system (10). When (1) $\alpha = \beta = \gamma = 1.1$, $\omega = 0.34$, $g = 1.5$; (2) $\alpha = \beta = 0.9$, $\gamma = 1.1$, $\omega = 0.62$, $g = 0.5$; (3) $\alpha = \beta = 0.8$, $\gamma = 1.1$, $\omega = 0.2807$, $g = 1$; (4) $\alpha = \beta = 0.7$, $\gamma = 1.1$, $\omega = 0.144$, $g = 3$; (5) $\alpha = \beta = 0.6$, $\gamma = 1.1$, $\omega = 0.0107$, $g = 6.6$; (6) $\alpha = \beta = 0.5$, $\gamma = 1.1$, $\omega = 0.001$, $g = 0.5$; (7) $\alpha = \beta = 0.4$, $\gamma = 1.1$, $\omega = 0.005$, $g = 0.5$; (8) $\alpha = \beta = 0.3$, $\gamma = 1.1$, $\omega = 0.0017$, $g = 1$, chaos synchronization is obtained. For saving space only the phase portraits and Poincaré maps of the fractional order synchronized systems and time history of error states plot for condition (8) are shown in Fig. 5. When $\gamma < 1$, no chaos exists in system (10). When α, β take the fractional number less than 0.3, no chaotic synchronization is found.

Part (2):

$Z = bgz_2$ where z_2 is the chaotic state of system (10). When (1) $\alpha = \beta = \gamma = 1.1$, $\omega = 0.34$, $g = 4$; (2) $\alpha = \beta = 0.9$, $\gamma = 1.1$, $\omega = 0.62$, $g = 3$; (3) $\alpha = \beta = 0.8$, $\gamma = 1.1$, $\omega = 0.2807$, $g = 13.5$; (4) $\alpha = \beta = 0.7$, $\gamma = 1.1$, $\omega = 0.144$, $g = 3$; (5) $\alpha = \beta = 0.6$, $\gamma = 1.1$, $\omega = 0.0107$, $g = 21.3$; (6) $\alpha = \beta = 0.5$, $\gamma = 1.1$, $\omega = 0.001$, $g = 3$; (7) $\alpha = \beta = 0.4$, $\gamma = 1.1$, $\omega = 0.005$, $g = 1$; (8) $\alpha = \beta = 0.3$, $\gamma = 1.1$, $\omega = 0.0017$, $g = 1$, chaos synchronization is obtained. For saving space only the phase portraits and Poincaré maps of the fractional order synchronized systems and time history of error states plot for condition (8) are shown in Fig. 6. When $\gamma < 1$, no chaos exists in system (10). When α, β take the fractional number less than 0.3, no chaotic synchronization is found.

Part (3):

$Z = bg\exp(z_1)$ where z_1 is the chaotic state of system (10). When (1) $\alpha = \beta = \gamma = 1.1$, $\omega = 0.34$, $g = 0.29$; (2) $\alpha = \beta = 0.9$, $\gamma = 1.1$, $\omega = 0.555$, $g = 1$; (3) $\alpha = \beta = 0.8$, $\gamma = 1.1$, $\omega = 0.2807$, $g = 1.8$; (4) $\alpha = \beta = 0.7$, $\gamma = 1.1$, $\omega = 0.144$, $g = 1$; (5) $\alpha = \beta = 0.6$, $\gamma = 1.1$, $\omega = 0.0107$, $g = 1$; (6) $\alpha = \beta = 0.5$, $\gamma = 1.1$, $\omega = 0.001$, $g = 3$; (7) $\alpha = \beta = 0.4$, $\gamma = 1.1$, $\omega = 0.005$, $g = 1.5$; (8) $\alpha = \beta = 0.3$, $\gamma = 1.1$, $\omega = 0.0017$, $g = 0.1$, chaos synchronization is obtained. For saving space only the phase portraits and Poincaré maps of the fractional order synchronized systems and time history of error states plot for condition (8) are shown in Fig. 7. When $\gamma < 1$, no chaos exists in system (10). When α, β take the fractional number less than 0.3, no chaotic synchronization is found.

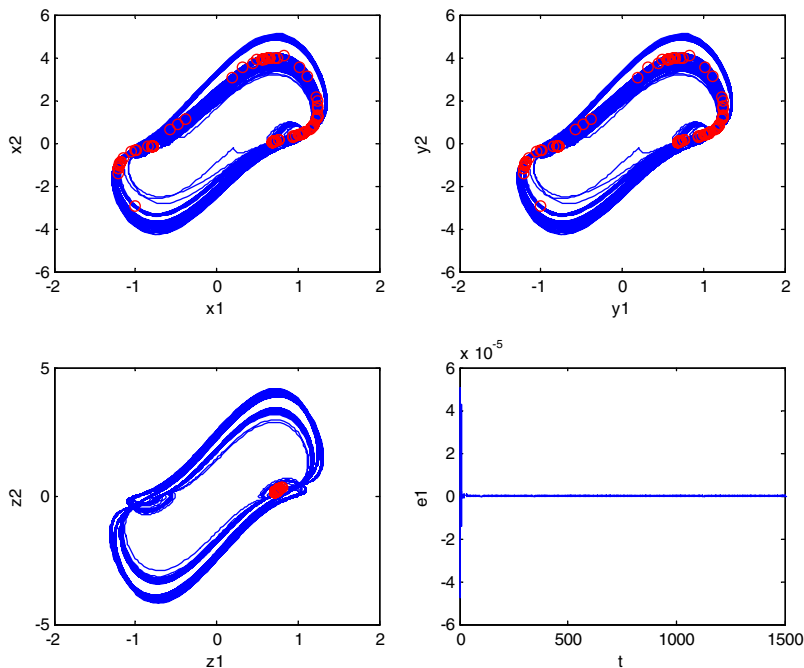


Fig. 4. Phase portraits and Poincaré maps of the synchronized fractional order systems and time history of states error for $Z = gz_2 \sin(\omega t)$ with order $\alpha = \beta = 0.7$, $\omega = 0.31812$, $g = 0.5$.

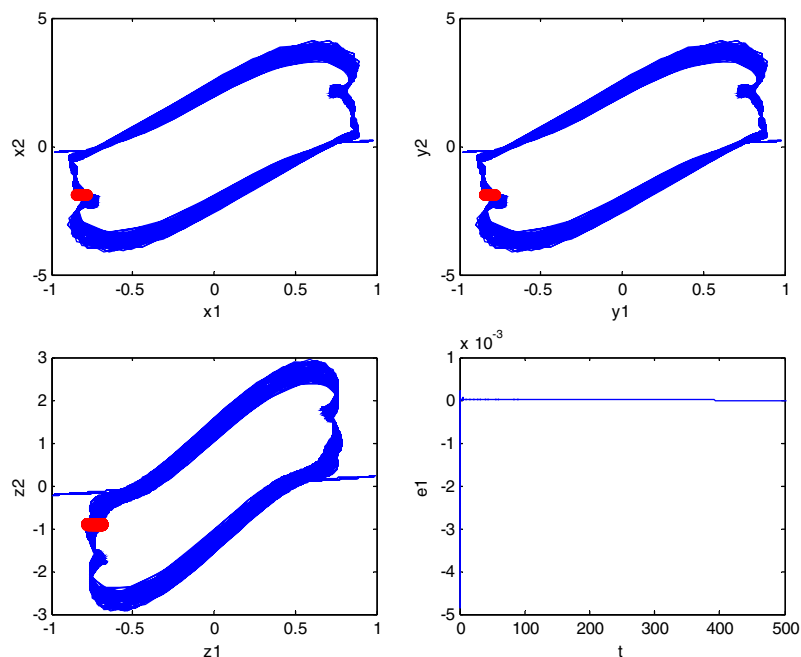


Fig. 5. Phase portraits and Poincaré maps of the synchronized fractional order systems and time history of states error for $Z = bgz_1$ with order $\alpha = \beta = 0.3$, $\gamma = 1.1$, $\omega = 0.0017$, $g = 1$.

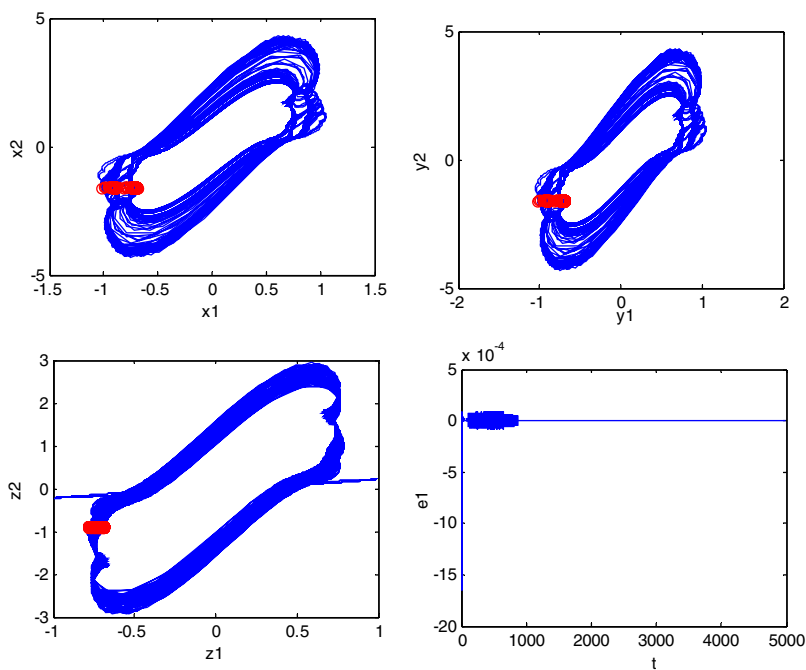


Fig. 6. Phase portraits and Poincaré maps of the synchronized fractional order systems and time history of states error for $Z = bgz_2$ with order $\alpha = \beta = 0.3$, $\gamma = 1.1$, $\omega = 0.0017$, $g = 1$.

Part (4):

$Z = b \exp(z_2)$ where z_2 is the chaotic state of system (10). When (1) $\alpha = \beta = \gamma = 1.1$, $\omega = 0.34$, $g = 0.2$; (2) $\alpha = \beta = 0.9$, $\gamma = 1.1$, $\omega = 0.555$, $g = 0.1$; (3) $\alpha = \beta = 0.8$, $\gamma = 1.1$, $\omega = 0.2807$, $g = 0.1$; (4) $\alpha = \beta = 0.7$, $\gamma = 1.1$, $\omega = 0.144$, $g = 1.9$; (5) $\alpha = \beta = 0.6$, $\gamma = 1.1$, $\omega = 0.0107$, $g = 2$; (6) $\alpha = \beta = 0.5$, $\gamma = 1.1$, $\omega = 0.001$, $g = 1$; (7) $\alpha = \beta = 0.4$, $\gamma = 1.1$, $\omega = 0.005$, $g = 4$; (8) $\alpha = \beta = 0.3$, $\gamma = 1.1$, $\omega = 0.0017$, $g = 0.1$, chaos synchronization is obtained. For saving space only the phase portraits and Poincaré maps of the fractional order synchronized systems and time history of error states plot for condition (8) are shown in Fig. 8. When $\gamma < 1$, no chaos exists in system (10). When α, β take the fractional number less than 0.3, no chaotic synchronization is found.

Part (5):

$Z = g \exp(z_1) \sin(\omega t)$ where z_1 is the chaotic state of system (10). When (1) $\alpha = \beta = \gamma = 1.1$, $\omega = 0.445$, $\omega_z = 0.34$, $g = 0.43$; (2) $\alpha = \beta = 0.9$, $\gamma = 1.1$, $\omega = 0.1275$, $\omega_z = 0.555$, $g = 0.1$; (3) $\alpha = \beta = 0.8$, $\gamma = 1.1$, $\omega = 0.1315$, $\omega_z = 0.2807$, $g = 0.1$; (4) $\alpha = \beta = 0.7$, $\gamma = 1.1$, $\omega = 0.31812$, $\omega_z = 0.144$, $g = 0.1$, chaos synchronization is obtained. For saving space only the phase portraits and Poincaré maps of the fractional order synchronized systems and time history of error states plot for condition (4) are shown in Fig. 9. When $\gamma < 1$, no chaos exists in system (10). When α, β take the fractional number less than 0.7, no chaotic synchronization is found.

Part (6):

$Z = g \exp(z_2) \sin(\omega t)$ where z_2 is the chaotic state of system (10). When (1) $\alpha = \beta = \gamma = 1.1$, $\omega = 0.445$, $\omega_z = 0.34$, $g = 0.1$; (2) $\alpha = \beta = 0.9$, $\gamma = 1.1$, $\omega = 0.1275$, $\omega_z = 0.555$, $g = 0.1$; (3) $\alpha = \beta = 0.8$, $\gamma = 1.1$, $\omega = 0.1315$, $\omega_z = 0.2807$, $g = 0.1$; (4) $\alpha = \beta = 0.7$, $\gamma = 1.1$, $\omega = 0.31812$, $\omega_z = 0.144$, $g = 0.1$, chaos synchronization is obtained. For saving space only the phase portraits and Poincaré maps of the fractional order synchronized systems and time history of error states plot for condition (4) are shown in Fig. 10. When $\gamma < 1$, no chaos exists in system (10). When α, β take the fractional number less than 0.7, no chaotic synchronization is found.

The ranges of g for various chaos synchronization cases are listed in Table 1.

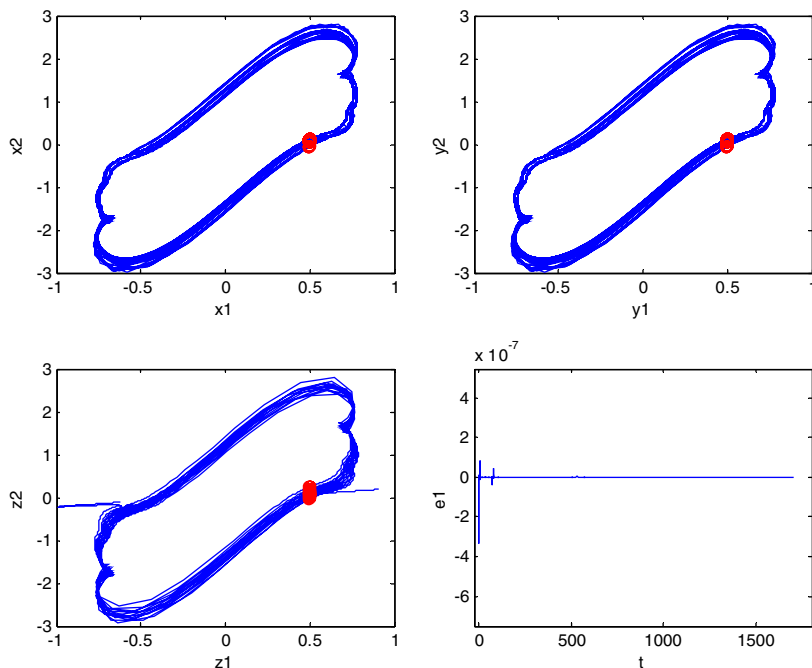


Fig. 7. Phase portraits and Poincaré maps of the synchronized fractional order systems and time history of states error for $Z = b \exp(z_1)$ with order $\alpha = \beta = 0.3$, $\gamma = 1.1$, $\omega = 0.0017$, $g = 0.1$.

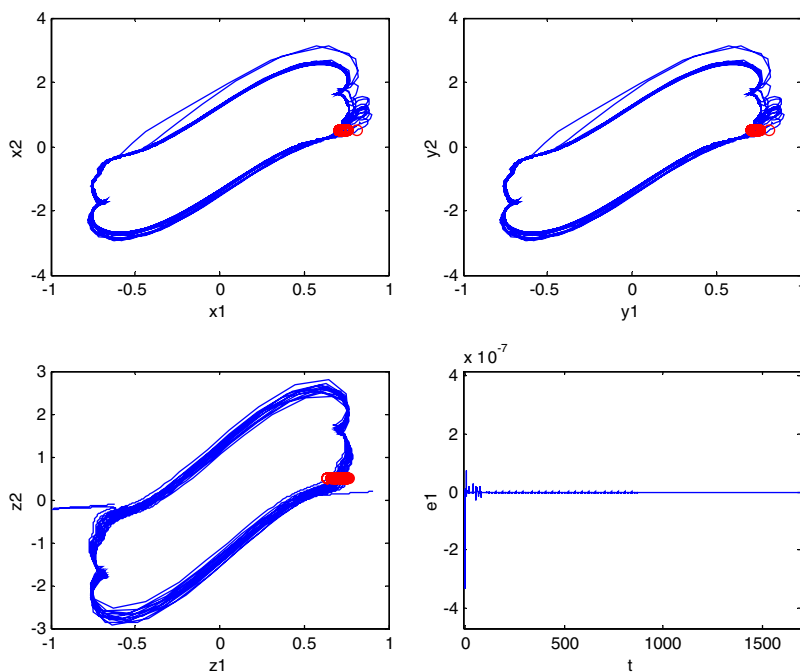


Fig. 8. Phase portraits and Poincaré maps of the synchronized fractional order systems and time history of states error for $Z = bge^{z_2}$ with order $\alpha = \beta = 0.3$, $\gamma = 1.1$, $\omega = 0.0017$, $g = 0.1$.

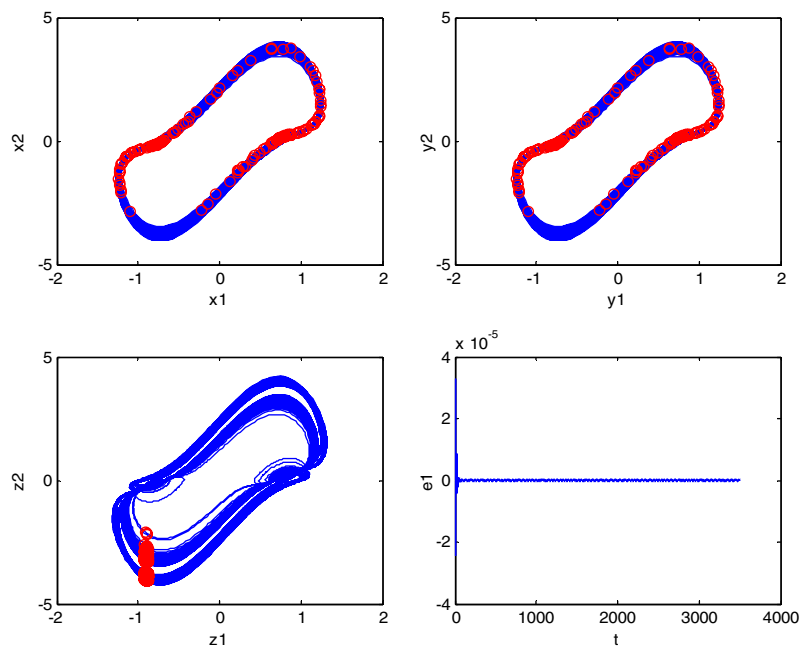


Fig. 9. Phase portraits and Poincaré maps of the synchronized fractional order systems and time history of states error for $Z = ge^{z_1} \sin(\omega t)$ with order $\alpha = \beta = 0.7$, $\gamma = 1.1$, $\omega = 0.31812$, $\omega_z = 0.144$, $g = 0.1$.

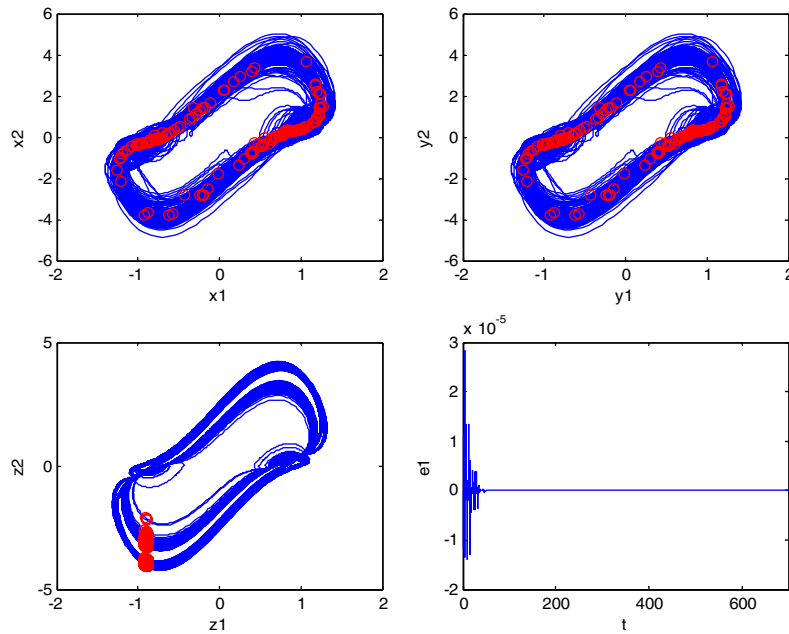


Fig. 10. Phase portraits and Poincaré maps of the synchronized fractional order systems and time history of states error for $Z = g e^{z_2} \sin(\omega t)$ with order $\alpha = \beta = 0.7$, $\gamma = 1.1$, $\omega = 0.31812$, $\omega_z = 0.144$, $g = 0.1$.

Table 1

The ranges of g for various chaos synchronization cases

	Part (1) $Z = b g z_1$	Part (2) $Z = b g z_2$	Part (3) $Z = g z_1 \sin(\omega t)$	Part (4) $Z = g z_2 \sin(\omega t)$
<i>Case 1: The third system is a nonautonomous system with two states, Eq. (9)</i>				
$\alpha = \beta = 1.1$	0.23–1.64	0.18–4.20	0.51–2.27	0.32–6.51
$\alpha = \beta = 1$	0.09–3.43	0.14–7.6	0.15–3.7	0.7–9.8
$\alpha = \beta = 0.9$	0.11–6.13	0.05–9.59	0.6–14.9	0.5–6.1
$\alpha = \beta = 0.8$	0.18–8.23	0.1–13.53	0.3–8.5	0.5–14.4
$\alpha = \beta = 0.7$	0.07–12.27	0.3–14.65	0.09–14.9	0.12–26.6
<i>Case 2: The third system is an autonomous system with three states, Eq. (10)</i>				
			$Z = b g \exp(z_1)$	$Z = b g \exp(z_2)$
$\alpha = \beta = \gamma = 1.1$	0.8–1.64	0.1–4.16	0.29–0.015	0.12–0.015
$\alpha = \beta = 0.9, \gamma = 1.1$	0.16–0.52	0.2–0.132	0.0001–1.36	0.001–0.18
$\alpha = \beta = 0.8, \gamma = 1.1$	0.1–11.6	0.07–13.5	0.0001–1.9	0.001–0.5
$\alpha = \beta = 0.7, \gamma = 1.1$	0.04–10.4	0.02–19.1	0.0001–3.6	0.001–1.9
$\alpha = \beta = 0.6, \gamma = 1.1$	0.06–6.6	0.05–21.3	0.0001–2.3	0.001–2.0
$\alpha = \beta = 0.5, \gamma = 1.1$	0.05–8.7	0.04–34.2	0.0001–3.2	0.001–3.6
$\alpha = \beta = 0.4, \gamma = 1.1$	0.06–10.1	0.1–38.2	0.02–3.7	0.001–4
$\alpha = \beta = 0.3, \gamma = 1.1$	0.06–11.9	0.04–50	0.01–4.7	0.01–5.5
	Part (5)		Part (6)	
	$Z = g \exp(z_1) \sin(\omega t)$		$Z = g \exp(z_2) \sin(\omega t)$	
$\alpha = \beta = \gamma = 1.1$	0.44–0.08		0.13–0.01	
$\alpha = \beta = 0.9, \gamma = 1.1$	0.03–2.14		0.003–0.33	
$\alpha = \beta = 0.8, \gamma = 1.1$	0.06–3.48		0.02–0.63	
$\alpha = \beta = 0.7, \gamma = 1.1$	0.05–5.22		0.007–1.93	

By the results of simulation, it is found that the chaos excited chaos synchronizations are obtained in Case 1 for lowest total fractional order $0.7 \times 2 = 1.4$, while synchronizations can be achieved in Case 2 for lowest total fractional order $0.3 \times 2 = 0.6$.

5. Conclusions

Chaos excited chaos synchronization of a generalized van der Pol system with integral and fractional order is studied. Synchronizations of two identical generalized van der Pol chaotic systems are obtained by replacing their corresponding exciting terms by the same function of chaotic states of a third nonautonomous or autonomous system. Numerical simulations, such as phase portraits, Poincaré maps and state error plots are given. It is found that chaos excited chaos synchronizations exist for the fractional order systems with the total fractional order both less than and more than the number of the states of the integer order generalized van der Pol system. Synchronizations are obtained in Case 1 for lowest total fractional order $0.7 \times 2 = 1.4$, while synchronizations can be achieved in Case 2 for lowest total fractional order $0.3 \times 2 = 0.6$.

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