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A Technique for Spectral Characterization of Some Regular Graphs

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Abstract

Illustrating by Odd graphs, a technique for testing distance-regularity among connected regular graphs in terms of their spectra is introduced. Following their known parametric characterizations, it eventually leads to spectral characterizations of some generalized Odd graphs among connected regular graphs.

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1 Introduction

Let $\Gamma = (V(\Gamma), E(\Gamma))$ be a connected regular graph with an adjacency matrix A. Since the rows and columns of A correspond to an arbitrary labelling of $V(\Gamma)$, it is clear that we shall be interested primarily in those properties of A which are invariant under permutations of rows and columns of A. Foremost among such properties are the spectral properties of A. If the distinct eigenvalues of A are $\theta_0 > \theta_1 > \cdots > \theta_{s-1}$ with multiplicities $m_0, m_1, m_2, \cdots, m_{s-1}$ respectively, we shall write

$$Spec(\Gamma) = (\theta_0^{m_0}, \theta_1^{m_1}, \theta_2^{m_2}, \cdots, \theta_{s-1}^{m_{s-1}})$$

and call it the *spectrum* of the graph Γ.

Some properties of graphs can be told easily from their spectra; for example: the number of distinct eigenvalues is at least one larger than its diameter; k-regular if and only if $(\theta_0, m_0) = (k, 1)$; bipartite if and only if $\theta_0 > \theta_1 > \cdots > \theta_{s-1}$ is symmetric with

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respect to 0, refer to [3] for details. However the spectral characterizations of graphs in general are very difficult. Indeed, spectral information usually pose strict restrictions over the structures of graphs with high regularity, and hence a lot of spectral characterizations of distance-regular graphs have been done under the assumption that the graphs are already distance-regular, while there are still a lot of non-distance-regular graphs cospectral to distance-regular graphs. Therefore, characterization of distance-regularity among connected regular graphs is an interesting area of investigation, ref to [1, p.369] for related backgrounds.

In addition to some already known technique such as *switching*, *interlacing* [4, p.85] for studying spectra of graphs, we will introduce another technique for dealing with spectral characterization of distance-regularity among connected regular graphs. In addition to its characteristic polynomial and minimal polynomial, a connected regular graph with an adjacency matrix A has another nonzero polynomial $q(x)$, called *Hoffman polynomial*, in terms of eigenvalues with a minimal degree such that all entries of $q(A)$ are equal. Under some additional conditions, systems of linear equations with coefficient matrices in terms of coefficients of $q(x)$ will be derived, and the distance-regularity of graphs under consideration depend on the fact that these systems of linear equations have unique solutions, i.e., those corresponding coefficient matrices have nonzero determinants. This procedure works well for bipartite distance-regular graphs with 4 distinct eigenvalues and for generalized Odd graphs. It seems quite reasonable to expect that this procedure might become standard to this type of problems, and work for some other classes of graphs with high degree of regularity.

Basic definitions such as distance-regular graphs, generalized Odd graphs are given in section 2, ref to [1, 4] for more details. The distance-regularity of those connected regular graphs with exactly four distinct eigenvalues, in particular $\{3q-3, 2q-3, q-3, -3\}$, are studied in section 3. Following illustrations for $H(3,2)$ and O_9, the main procedure is presented in section 4.

2 Preliminary

If $\Gamma = (V(\Gamma), E(\Gamma))$ is a connected graph with an adjacency matrix A and with $Spec(\Gamma) = (\theta_0^{m_0}, \theta_1^{m_1}, \theta_2^{m_2}, \cdots, \theta_{s-1}^{m_{s-1}})$, then $f(x) = \prod_{i=0}^{s-1}(x-\theta_i)^{m_i}$, $m(x) = \prod_{i=0}^{s-1}(x-\theta_i)$ are called the characteristic and the minimal polynomial of Γ respectively. It is well known that $f(A) = m(A) = 0$. However, in case Γ is connected and k-regular, then $(\theta_0, m_0) = (k, 1)$, and the polynomial $q(x) = \prod_{i=1}^{s-1}(x-\theta_i)$, called the *Hoffman polynomial* of Γ, is of another interest. The significance of $q(x)$ lies on the fact that it is the unique polynomial of the smallest degree such that $q(A) = \frac{q(k)}{|V(G)|}J$ where J is the all-one matrix. Note that it is the polynomial $\frac{|V(G)|}{q(k)}q(x)$ called the Hoffman polynomial of Γ in the literature.

Those graphs with high degree of regularity are interesting to us. Let

$$\Gamma_i(x) = \{y | y \in V(\Gamma) \text{ and } \partial(x,y) = i\}$$

where $\partial(x,y)$ denotes the distance between x and y in Γ. Let

$$\begin{aligned} c_i(x,y) &= |\Gamma_1(x) \cap \Gamma_{i-1}(y)|, \\ a_i(x,y) &= |\Gamma_1(x) \cap \Gamma_i(y)|, \\ b_i(x,y) &= |\Gamma_1(x) \cap \Gamma_{i+1}(y)| \end{aligned}$$

for $x, y \in V(\Gamma)$ at distance i, $0 \le i \le d$. In particular, $c_0(x,x) = a_0(x,x) = 0$, $c_1(x,y) = 1$ if $\partial(x,y) = 1$, $b_0(x,x) = k$, $b_d(x,y) = 0$ if $\partial(x,y) = d$, and $a_i(x,y) + b_i(x,y) + c_i(x,y) = k$ if $\partial(x,y) = i$. The graph Γ is called *distance-regular* if $c_i(x,y)$, $a_i(x,y)$ and $b_i(x,y)$ depend not on the particular vertices x, y we choose, but only on the distance $i = \partial(x,y)$ between them. In this case, the common values are denoted by c_i, a_i and b_i respectively, and $\{b_0, b_1, b_2, \cdots, b_{d-1}; c_1, c_2, c_3, \cdots, c_d\}$, or

$$\begin{bmatrix} c_0 & c_1 & c_2 & \cdots & c_d \\ a_0 & a_1 & a_2 & \cdots & a_d \\ b_0 & b_1 & b_2 & \cdots & b_d \end{bmatrix}$$

is called the *intersection array* of the distance-regular graph Γ. Clearly, Γ is bipartite if and only if $a_0 = a_1 = a_2 = \cdots = a_d = 0$, and hence it has no odd cycles. In this paper, we are interested in those with $a_0 = a_1 = \cdots = a_{d-1} = 0$ and $a_d \neq 0$, they are called *generalized Odd graphs* [4, p.145]. Odd graphs as shown below are among them.

The following two families of distance-regular graphs will be treated in this paper. Let F be a set of q symbols. The *Hamming graph* $H(n,q)$ is defined over F^n such that two vertices $(x_1, \cdots, x_n)$, $(y_1, \cdots, y_n) \in F^n$ are adjacent if and only if the Hamming distance between them is exactly 1. It is known that $H(n,q)$ is a distance-regular graph with intersection array $\{n(q-1), (n-1)(q-1), (n-2)(q-1), \cdots, (q-1); 1, 2, 3, \cdots, n\}$ and its spectrum is given by $\theta_i = (q-1)n - qi$ with $m_i = \binom{n}{i}(q-1)^i$ for $0 \le i \le n$. In particular, if $q = 2$, then $H(n,2)$ is bipartite with intersection array $\{n, n-1, n-2, \cdots, 1; 1, 2, \cdots, n-1, n\}$ and with spectrum $(n^{\binom{n}{0}}, (n-2)^{\binom{n}{1}}, (n-4)^{\binom{n}{2}}, \cdots, -(n-2)^{\binom{n}{n-1}}, -n^{\binom{n}{n}})$.

Others are Odd graphs O_d, $d \ge 2$, defined over the $(d-1)$-subsets of $\{1, 2, \ldots, 2d-1\}$ such that two vertices are adjacent if and only if their corresponding subsets are disjoint. The small Odd graphs are the triangle K_3 $(d = 2)$ and the Petersen graph $(d = 3)$. In general, the Odd graphs O_d are distance-regular graphs of diameter $d-1$ with intersection array

(1) if $d = 2l$

$$\begin{bmatrix} 0 & 1 & 1 & 2 & 2 & \cdots & l-1 & l-1 & l \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & l \\ 2l & 2l-1 & 2l-1 & 2l-2 & 2l-2 & \cdots & l+1 & l+1 & 0 \end{bmatrix}$$

(2) if $d = 2l+1$

$$\begin{bmatrix} 0 & 1 & 1 & 2 & 2 & \cdots & l & l \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & l+1 \\ 2l+1 & 2l & 2l & 2l-1 & 2l- & \cdots & l+1 & 0 \end{bmatrix}$$

The eigenvalues of O_d are the integers $\theta_i = (-1)^i(d-i)$ with multiplicity $m_i = \binom{2d-1}{i} - \binom{2d-1}{i-1}$ respectively for $0 \leq i \leq d-1$.

Some properties of graphs can be transformed into their cospectral mates through some counting formulae. If Γ is a connected graph with an adjacency matrix A and with $Spec(\Gamma) = Spec(G) = (\theta_0^{m_0}, \theta_1^{m_1}, \cdots, \theta_d^{m_d})$, where G is a graph with some interesting properties, then $\mathrm{Tr}(A^j) = \sum_{i=0}^{d} m_i\theta_i^j$ is the number of closed walks of length j in both Γ and G. For example, if Γ has no cycle of length j, then $\mathrm{Tr}(A_G^j) = \sum_{i=0}^{d} m_i\theta_i^j = 0$ where A_G is an adjacency matrix of G. It follows that $\mathrm{Tr}(A^j) = 0$ too, and hence Γ has no closed walk j either. The following lemma follows from the above observation immediately, which plays a significant role in later spectral characterization of generalized Odd graphs.

Lemma 2.1 *Let Γ be a connected regular graph with an adjacency matrix A. If Γ is cospectral with a generalized Odd graph of diameter d, then*

1) $A_{xx}^{2i+1} = 0$ *for* $i \leq d-2$,

2) $A_{xy}^{2i+1-j} = 0$ *for* $x, y \in V(\Gamma)$ *at distance* $j \leq i$,

3) $a_i(x, y) = 0$ *for* $x, y \in V(\Gamma)$ *at distance* $i \leq d-1$.

3 Spectra of Graphs and Distance-regularity

It is well known that any distance-regular graph with diameter d has exactly $d+1$ distinct eigenvalues [3, p.10], and conversely any connected regular graph with 3 distinct eigenvalues must be strongly regular *i.e.*, distance-regular with diameter 2. It would be of interesting to see those connected regular graphs of diameter d with exactly $d+1$ distinct eigenvalues. However, distance-regularity can not be told simply by comparing its diameter and number of distinct eigenvalues as shown in the following counter-examples. In this section, some sufficient conditions for graphs with four distinct eigenvalues will be considered for this purpose.

The following are two examples of connected regular graphs with diameter d and with exactly $d+1$ distinct eigenvalues for $d = 3, 4$ respectively while neither of them is distance-regular.

(1) Let Γ be the Hamming graph $H(3,2)$ with edges as labelled in figure (i), then $Spec(\Gamma) = (3^1, 1^3, -1^3, -3^1)$, and hence the spectrum of its line graph $L(\Gamma)$, figure (ii), is given by $Spec(L(\Gamma)) = (4^1, 2^3, 0^3, -2^5)$. Certainly, $L(\Gamma)$ is connected, 4-regular of diameter 3, and with exactly 4 distinct eigenvalues. However, it is not distance-regular by comparing the sizes of common neighborhoods for vertices $\{1, 8\}$ and of $\{5, 6\}$ at distance 2.

(2) The graph Γ given in figure (iii), see also [4, p.263], is connected regular of diameter 4, with $Spec(\Gamma) = (4^1, 2^4, 0^6, -2^4, -4^1)$ of 5 distinct eigenvalues, cospectral with $H(4,2)$. Both of their adjacency matrices satisfy the equation $A(A-2)(A+2)(A+4) = 24J$. Note that this graph is bipartite, and it has one more eigenvalue -4 than that of $L(H(3,2))$.

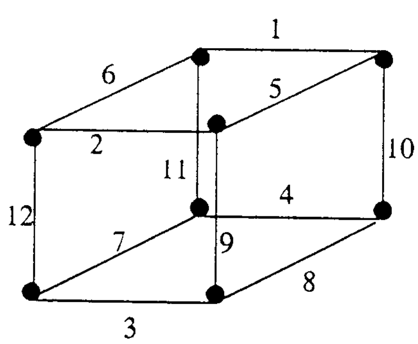

figure (i)

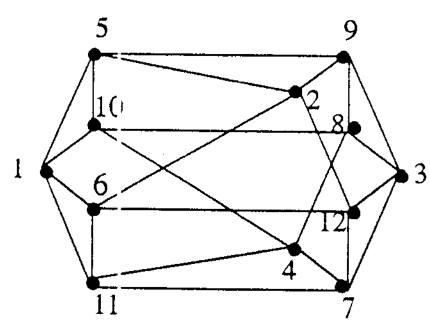

figure (ii)

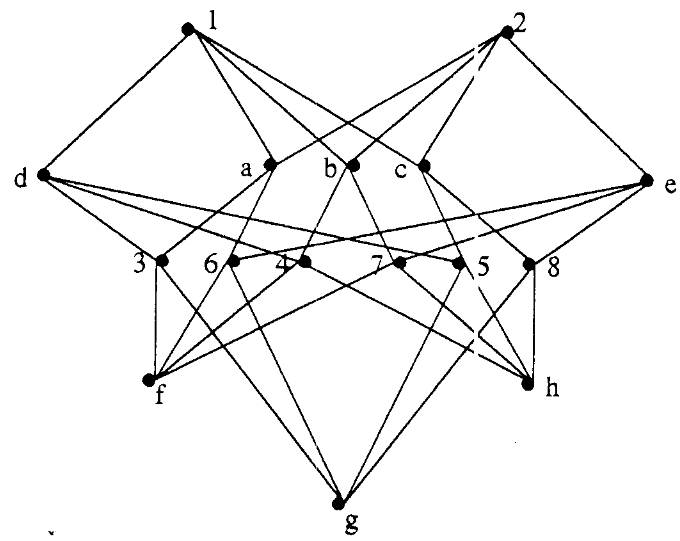

figure (iii)

The distance-regularity of connected regular bipartite graphs with 4 distinct eigenvalues is given in the following. In terms of Hoffman polynomial, it will be proved by taking full advantage that bipartite graphs have no odd cycles.

Theorem 3.1 *Let Γ be a connected k-regular bipartite graph with n vertices and with exactly 4 distinct eigenvalues $\pm k, \pm\theta$ $(\theta > 0)$. Then Γ is a distance-regular graph of diameter 3 with intersection array*

$$\{k, k-1, k - \tfrac{2(k^2-\theta^2)}{n}; 1, \tfrac{2(k^2-\theta^2)}{n}, k\}.$$

Proof. Let A be an adjacency matrix of Γ, then its Hoffman polynomial gives

$$A^3 + kA^2 - \theta^2 A - k\theta^2 I = \tfrac{2k(k^2-\theta^2)}{n} J.$$

Since Γ is bipartite, there is no odd cycle in Γ and hence $A(x,x) = A^3(x,x) = 0$, *i.e.*, $a_i(x,y) = 0$ for all $x, y \in V(\Gamma)$ at distance i. Suppose $x, y \in V(\Gamma)$ at distance 2, then $A^3(x,y) = 0$. Since $kA^2(x,y) = \frac{2k(k^2-\theta^2)}{n}$, it follows that $c_2(x,y) = A^2(x,y) = \frac{2(k^2-\theta^2)}{n}$, and $b_2(x,y) = k - c_2(x,y) = k - \frac{2(k^2-\theta^2)}{n}$. Suppose $x, y \in V(\Gamma)$ at distance 3, then $A^3(x,y) = \frac{2k(k^2-\theta^2)}{n}$. On the other hand, $A^3(x,y) = \sum_{z\in\Gamma_2(x)\cap\Gamma_1(y)} A^2(x,z) = \frac{2(k^2-\theta^2)}{n} c_3(x,y)$, it follows that $c_3(x,y) = k$ and hence Γ is distance-regular with intersection array as required. QED

Among such cases, the set $\{3q-3, 2q-3, q-3, -3\}$ attracts us most because it can be realized as the set of distinct eigenvalues of the Hamming graph $H(3,q)$. Note that it is bipartite if $q = 2$. Hoffman [9] showed that the set $\{3, 1, -1, -3\}$ (i.e., $q = 2$) is only realized by $H(3, 2)$ as its set of distinct eigenvalues among connected regular graphs of 8 vertices. This can be improved in the following corollary, and another approach will also be considered in section 4.

Corollary 3.2 *If Γ is a connected regular graph with distinct eigenvalues $\{3, 1, -1, -3\}$, then Γ must be isomorphic to $H(3, 2)$.*

Proof. Let $k = 3$ and $\theta = 1$ in the previous theorem, then $|V(\Gamma)|$ must be a divisor of 16 and hence $|V(\Gamma)| = 8$ or 16. In case $|V(\Gamma)| = 8$, then Γ must be distance-regular with intersection array $\{3, 2, 1; 1, 2, 3\}$ and hence Γ is isomorphic to $H(3, 2)$. The case $|V(\Gamma)| = 16$ can be ruled out easily by noting that the corresponding intersection array $\{2, 1, 1; 1, 2, 2\}$ is not feasible at all. QED

For those cases $q \geq 4$, similar technique can be applied to derive its diameter if the number of vertices is large enough. It is worth mentioning here that Lasker [12] showed that a connected regular graph Γ with distinct eigenvalues $\{3q-3, 2q-3, q-3, -3\}$, $q \geq 8$, with q^3 vertices and with $|\Gamma_2(x)| = 3(q-1)^2$ for each $x \in v(\Gamma)$ is isomorphic to $H(3, q)$.

Theorem 3.3 *Let Γ be a connected regular graph with 4 distinct eigenvalues $3q-3, 2q-3, q-3, -3$, and with at least $\frac{(6q^2-9q+10)+\sqrt{(6q^2-9q-10)^2-24q^3}}{2}$ vertices, then the diameter of Γ is 3.*

Proof. Let $|V(\Gamma)| = n$, and A be an adjacency matrix of Γ, then

$$A^3 - (3q-9)A^2 + (2q^2 - 18q + 27)A + (6q^2 - 27q + 27)I = \frac{6q^3}{n}J \qquad (*)$$

in terms of its Hoffman polynomial, and hence the diameter of Γ is at most 3. Moreover $(A^i)_{xx}$ for $i = 2, 3$ can be evaluated as follows: $(A^2)_{xx} = \sum_{y \in \Gamma_1(x)} A_{xy} = 3(q-1)$, and hence

$$\begin{aligned}
(A^3)_{xx} &= \sum_{y \in \Gamma_1(x)} A^2_{xy} \\
&= \frac{6q^3}{n} + (3q-9)A^2_{xx} - (2q^2 - 18q + 27)A_{xx} - (6q^2 - 27q + 27)I_{xx} \\
&= \frac{6q^3}{n} + 3q(q-3).
\end{aligned}$$

For any $x \in V(\Gamma)$, since $(A^2J)_{xx} = \sum_{y \in V(\Gamma)} A^2_{xy} = (3q-3)^2$,

$$\begin{aligned}
\sum_{y \in \Gamma_2(x)} A^2_{xy} &= \sum_{y \in V(\Gamma)} A^2_{xy} - A^2_{xy} - \sum_{y \in \Gamma_3(x)} A^2_{xy} \\
&= 6q^2 - 12q + 12 - \frac{6q^3}{n} \geq |\Gamma_2(x)|.
\end{aligned}$$

It follows that $1 + |\Gamma_1(x)| + |\Gamma_2(x)| \leq 1 + (3q-3) + (6q^2 - 12q + 12 - \frac{6q^3}{n}) < n$ under the assumption. Hence the diameter of Γ is 3. QED

Indeed, a necessary and sufficient condition for connected regular graphs with $Spec(G) = (k^1, \theta_1^{m_1}, \theta_2^{m_2}, \theta_3^{m_3})$ to be distance-regular was given in [8] that $|\Gamma_2(x)| = \frac{k(k-1-\lambda)^2}{(k-\lambda)(\lambda+q_2)-k-q_1+q_0}$ for each vertex x of G, where $q(x) = (x-\theta_1)(x-\theta_2)(x-\theta_3) = x^3 + q_2x^2 + q_1x + q_0$, and $\lambda = (k^3 + m_1\theta_1^3 + m_2\theta_2^3 + m_3\theta_2^3)/k(1 + m_1 + m_2 + m_3)$. The condition on girth can also be used in spectral characterization of distance-regular graphs. For example, Brouwer and Haemers [6] showed that a connected regular graph with the spectrum of a distance-regular graph of diameter d and with girth at least $2d-1$ is such a graph.

4 A procedure for testing distance-regularity

Motivated from the roles played by Hoffman polynomials in the proofs of Theorem 3.1, we will introduce in this section a procedure for testing distance-regularity among connected regular graphs in terms of their Hoffman polynomials. We will first treat $H(3,2)$ directly as a preliminary illustration, followed by an explanation why it does not work for $H(n,2)$, $n \geq 4$. We then treat the Odd graph O_9 in much more detail as another illustration. A general procedure will then be concluded for such purpose. This procedure has been successfully applied to those connected regular graphs which are cospectral with generalized Odd graphs.

Let Γ be a regular connected graph with an adjacency matrix A, and with spectrum $(3^1, 1^3, -1^3, -3^1)$ as that of $H(3,2)$, then $q(A) = A^3 + 3A^2 - A - 3I = 6J$. For vertices x, y at distance 2, substituting $A^3(x,y) = 0$ into $q(A) = 6J$, we have the equation $3A^3(x,y) = 6$ by a straightforward calculation, and hence $A^2(x,y) = 2$. Furthermore,

$$A^2(x,y) = \sum_{z \in \Gamma_1(x) \cap \Gamma_1(y)} A^1(z\ y) = c_2(x,y)$$

since $A^1(z,y) = 1$ in case $\partial(z,y) = 1$. It follows that $c_2 = c_2(x,y) = 2$ and $b_2 = b_2(x,y) = 1$ whenever $\partial(x,y) = 2$. For vertices x, y at distance 3, then $A^3(x,y) = 6$ similarly. Furthermore,

$$A^3(x,y) = \sum_{z \in \Gamma_1(x) \cap \Gamma_2(y)} A^2(z,y) = 1 \cdot c_3(x,y)$$

since $A^2(z,y) = 2$ if $\partial(z,y) = 2$ as just shown. It follows that $c_3 = c_3(x,y) = 3$ and $b_3 = b_3(x,y) = 0$ whenever $\partial(x,y) = 3$. Up to this point, we conclude that Γ is distance-regular with intersection array $\{3,2,1;1,2,3\}$ and hence Γ is isomorphic to $H(3,2)$. This provides an alternate approach to Corollary 3.2.

However, this procedure does not work for bipartite graphs $H(n,2)$ with $n \geq 4$ due to the fact that their eigenvalues are symmetric with respect to 0. For a connected regular graph with an adjacency matrix A and with spectrum $(4^1, 2^4, 0^6, -2^4, -4^1)$ as that of $H(4,2)$, then $q(A) = A^4 + 4A^3 - 4A^2 - 16A = 24J$. For vertices x, y at distance 2, substituting $A^5(x,y) = A^3(x,y) = 0$ into $q(A) = 24 \cdot J$ and $A \cdot q(A) = 24 \cdot 4 \cdot J$ respectively, we have the system of linear equations

$$\begin{bmatrix} 4 & -16 \\ 1 & -4 \end{bmatrix} \begin{bmatrix} A^4(x,y) \\ A^2(x,y) \end{bmatrix} = 24 \begin{bmatrix} 4^1 \\ 4^0 \end{bmatrix}$$

with determinant 0 for its coefficient matrix. No information regarding distance-regularity of Γ can be derived from it. Indeed, a counter-example can be found in [9]. Similar situation occurs for graphs cospectral with $H(5,2)$.

We now move to another illustration of this technique in terms of Odd graph O_9 in detail. Clearly, O_9 is 9-regular of diameter 8 and with

$$Spec(O_9) = (\ 9^1 \quad 7^{119} \quad 5^{1700} \quad 3^{6188} \quad 1^{4862} \quad -2^{7072} \quad -4^{3808} \quad -6^{544} \quad -8^{16})\ .$$

Let Γ be a connected regular graph with an adjacency matrix A and with spectrum as given above, and let

$$q(x) = (x-1)(x+2)(x-3)(x+4)(x-5)(x+6)(x-7)(x+8).$$

It follows that

$$\begin{aligned} q(A) &= A^8 + 4A^7 - 94A^6 - 296A^5 + 2609A^4 + 5716A^3 \\ &\quad -22676A^2 - 25584A + 40320I \\ &= 576J. \end{aligned}$$

As mentioned before, $a_i = a_i(x,y) = 0$ if $\partial(x,y) = i \leq 7$. To determine $c_i(x,y)$ with $\partial(x,y) = i$ for Γ, we shall compute $A^i(x,y)$ with $\partial(x,y) = i$ in two ways by either

1) solving a system of linear equations obtained from $q(A) = 576J$, or

2) applying the fact that

$$\begin{aligned} A^i(x,y) &= \sum_{z \in \Gamma_1(x) \cap \Gamma_{i-1}(y)} A^{i-1}(z,y) \\ &\quad + \sum_{z \in \Gamma_1(x) \cap \Gamma_i(y)} A^{i-1}(z,y) \\ &\quad + \sum_{z \in \Gamma_1(x) \cap \Gamma_{i+1}(y)} A^{i-1}(z,y) \end{aligned}$$

inductively.

In order to derive c_2, let us start from vertices x, y at distance 2, substituting $A^7(x,y) = A^5(x,y) = A^3(x,y) = 0$ into $A^i \cdot q(A) = 576 \cdot 9^i \cdot J$ for $0 \le i \le 5$, we have the system of linear equations

$$\begin{bmatrix} 4 & -296 & 5716 & -25584 & 0 & 0 \\ 1 & -94 & 2609 & -22676 & 40320 & 0 \\ 0 & 4 & -296 & 5716 & -25584 & 0 \\ 0 & 1 & -94 & 2609 & -22676 & 40320 \\ 0 & 0 & 4 & -296 & 5716 & -25584 \\ 0 & 0 & 1 & -94 & 2609 & -22676 \end{bmatrix} \begin{bmatrix} A^{12}(x,y) \\ A^{10}(x,y) \\ A^{8}(x,y) \\ A^{6}(x,y) \\ A^{4}(x,y) \\ A^{2}(x,y) \end{bmatrix} = 576 \begin{bmatrix} 9^5 \\ 9^4 \\ 9^3 \\ 9^2 \\ 9^1 \\ 9^0 \end{bmatrix}$$

and hence $A^2(x,y) = 1$ by a straightforward calculation. On the other hand,

$$A^2(x,y) = \sum_{z \in \Gamma_1(x) \cap \Gamma_1(y)} A^1(z,y) = c_2(x,y),$$

since $A^1(z,y) = 1$ if $\partial(z,y) = 1$. It follows that $c_2 = c_2(x,y) = 1$ and $b_2 = b_2(x,y) = 8$ whenever $\partial(x,y) = 2$.

Following the fact that $c_2 = 1$, we can treat similarly for vertices x, y at distance 3 to derive c_3. Let x, y be two vertices at distance 3, substituting $A^8(x,y) = A^6(x,y) = A^4(x,y) = 0$ into $A^i \cdot q(A) = 576 \cdot 9^i \cdot J$ for $0 \le i \le 4$, we have the system of linear equations

$$\begin{bmatrix} 4 & -296 & 5716 & -25584 & 0 \\ 1 & -94 & 2609 & -22676 & 40320 \\ 0 & 4 & -296 & 5716 & -25584 \\ 0 & 1 & -94 & 2609 & -22676 \\ 0 & 0 & 4 & -296 & 5716 \end{bmatrix} \begin{bmatrix} A^{11}(x,y) \\ A^{9}(x,y) \\ A^{7}(x,y) \\ A^{5}(x,y) \\ A^{3}(x,y) \end{bmatrix} = 576 \begin{bmatrix} 9^4 \\ 9^3 \\ 9^2 \\ 9^1 \\ 9^0 \end{bmatrix}$$

and hence $A^3(x,y) = 2$ by a straightforward calculation. On the other hand,

$$A^3(x,y) = \sum_{z \in \Gamma_1(x) \cap \Gamma_2(y)} A^2(z,y) = 1 \cdot c_3(x,y),$$

since $A^2(z,y) = 1$ if $\partial(z,y) = 2$ as just shown. It follows that $c_3 = c_3(x,y) = 1$ and $b_3 = b_3(x,y) = 7$ whenever $\partial(x,y) = 3$.

As a matter of fact, the above procedure can be done recursively for vertices x, y at distance i up to 8 to determine $c_i(x,y)$ in terms of $c_2, c_3, \cdots, c_{i-1}$. All details will be included in the following for illustration purpose.

(1) Let x, y be two vertices at distance 4, substituting $A^7(x,y) = A^5(x,y) = 0$ into

$A^i \cdot q(A) = 576 \cdot q^i \cdot J$ for $0 \le i \le 3$, we have the system of linear equations

$$\begin{bmatrix} 4 & -296 & 5716 & -25584 \\ 1 & -94 & 2609 & -22676 \\ 0 & 4 & -296 & 5716 \\ 0 & 1 & -94 & 2609 \end{bmatrix} \begin{bmatrix} A^{10}(x,y) \\ A^8(x,y) \\ A^6(x,y) \\ A^4(x,y) \end{bmatrix} = 576 \begin{bmatrix} 9^3 \\ 9^2 \\ 9^1 \\ 9^0 \end{bmatrix}$$

and hence $A^4(x,y) = 4$ by a straightforward calculation. Furthermore,

$$A^4(x,y) = \sum_{z \in \Gamma_1(x) \cap \Gamma_3(y)} A^3(z,y) = 2 \cdot c_4(x,y),$$

since $A^3(z,y) = 2$ if $\partial(z,y) = 3$ as shown. It follows that $c_4 = c_4(x,y) = 2$ and $b_4 = b_4(x,y) = 7$ whenever $\partial(x,y) = 4$.

(2) Let x, y be two vertices at distance 5, substituting $A^8(x,y) = A^6(x,y) = 0$ into $A^i \cdot q(A) = 576 \cdot q^i \cdot J$ for $0 \le i \le 2$, we have the system of linear equations

$$\begin{bmatrix} 4 & -296 & 5716 \\ 1 & -94 & 2609 \\ 0 & 4 & -296 \end{bmatrix} \begin{bmatrix} A^9(x,y) \\ A^7(x,y) \\ A^5(x,y) \end{bmatrix} = 576 \begin{bmatrix} 9^2 \\ 9^1 \\ 9^0 \end{bmatrix}$$

and hence $A^5(x,y) = 12$ by a straightforward calculation. Furthermore,

$$A^5(x,y) = \sum_{z \in \Gamma_1(x) \cap \Gamma_4(y)} A^4(z,y) = 4 \cdot c_5(x,y),$$

since $A^4(z,y) = 4$ if $\partial(z,y) = 4$ as shown in (1). It follows that $c_5 = c_5(x,y) = 3$ and $b_5 = b_5(x,y) = 6$ whenever $\partial(x,y) = 5$.

(3) Let x, y be two vertices at distance 6, substituting $A^7(x,y) = 0$ into $A^i \cdot q(A) = 576 \cdot 9^i \cdot J$ for $0 \le i \le 1$, we have the system of linear equations

$$\begin{bmatrix} 4 & -296 \\ 1 & -94 \end{bmatrix} \begin{bmatrix} A^8(x,y) \\ A^6(x,y) \end{bmatrix} = 576 \begin{bmatrix} 9^1 \\ 9^0 \end{bmatrix}$$

and hence $A^6(x,y) = 12$ by a straightforward calculation. Furthermore,

$$A^6(x,y) = \sum_{z \in \Gamma_1(x) \cap \Gamma_5(y)} A^5(z,y) = 12 \cdot c_6(x,y),$$

since $A^4(z,y) = 4$ if $\partial(z,y) = 5$ as shown in (2). It follows that $c_6 = c_6(x,y) = 3$ and $b_6 = b_6(x,y) = 6$ whenever $\partial(x,y) = 6$.

(4) Let x, y be two vertices at distance 7, substituting $A^8(x,y) = 0$ into $A^i \cdot q(A) = 576 \cdot q^i \cdot J$, we have the system of linear equat ons

$$[4][A^7(x,y)] = 576[9^0]$$

and hence $A^7(x,y) = 144$ by a straightforward calculation. Furthermore,

$$A^7(x,y) = \sum_{z \in \Gamma_1(x) \cap \Gamma_6(y)} A^6(z,y) = 36 \cdot c_7(x,y),$$

since $A^6(z,y) = 36$ if $\partial(z,y) = 6$ as shown in (3). It follows that $c_7 = c_7(x,y) = 4$ and $b_7 = b_7(x,y) = 5$ whenever $\partial(x,y) = 7$.

(5) Let x, y be two vertices at distance 8, then $A^8(x,y) = 576$. Furthermore,

$$A^8(x,y) = \sum_{z \in \Gamma_1(x) \cap \Gamma_7(y)} A^7(z,y) = 144 \cdot c_8(x,y),$$

since $A^7(z,y) = 144$ if $\partial(z,y) = 7$ as shown in (4). It follows that $c_8 = c_8(x,y) = 4$ and hence $a_8 = a_8(x,y) = 5$ whenever $\partial(x,y) = 8$.

Hence Γ is a distance-regular graph with the same intersection array $\{9,8,8,7,7,6,6,5;1,1,2,2,3,3,4,4\}$ as that of O_9, it follows that Γ is isomorphic to O_9 by its known parametric characterization [4, p.260].

The argument works perfectly for Odd graph O_9 totally depends on the fact that the determinants of the coefficient matrix

$$\begin{bmatrix} 4 & -296 & 5716 & -25584 & 0 & 0 \\ 1 & -94 & 2609 & -22676 & 40320 & 0 \\ 0 & 4 & -296 & 5716 & -25584 & 0 \\ 0 & 1 & -94 & 2609 & -22676 & 40320 \\ 0 & 0 & 4 & -296 & 5716 & -25584 \\ 0 & 0 & 1 & -94 & 2609 & -22676 \end{bmatrix}$$

and five of its principal minors are nonzero. To be more precise, and for later systematic development, let $q(x) = \sum_{i=0}^{8} a(8,i)x^i$, let

$$M_{8,0} = \begin{bmatrix} a(8,7) & a(8,5) & a(8,3) & a(8,1) & 0 & 0 \\ a(8,8) & a(8,6) & a(8,4) & a(8,2) & a(8,0) & 0 \\ 0 & a(8,7) & a(8,5) & a(8,3) & a(8,1) & 0 \\ 0 & a(8,8) & a(8,6) & a(8,4) & a(8,2) & a(8,0) \\ 0 & 0 & a(8,7) & a(8,5) & a(8,3) & a(8,1) \\ 0 & 0 & a(8,8) & a(8,6) & a(8,4) & a(8,2) \end{bmatrix},$$

and $M_{8,i}$ be obtained from $M_{8,0}$ by deleting the last i rows as well as the last i columns, $1 \leq i \leq 5$. The key to the success of the previous arguments lie on the facts that $\det(M_{8,i}) \neq 0$ for $0 \leq i \leq 5$. The following data for Odd graph O_{12} were kept record here for reference purpose:

$$O_{12}: \; x^{11} + 6x^{10} - 235x^9 - 1230x^8 + 19023x^7 + 83538x^6 - 636385x^5$$
$$-2206770x^4 + 8245276x^3 + 20268456x^2 - 29400480x - 39916800$$

$$M_{11,0} =$$

$$\begin{bmatrix} 6 & -1230 & 83538 & -2206770 & 20268456 & -39916800 & 0 & 0 & 0 \\ 1 & -235 & 19023 & -636385 & 8245276 & -29400480 & 0 & 0 & 0 \\ 0 & 6 & -1230 & 83538 & -2206770 & 20268456 & -39916800 & 0 & 0 \\ 0 & 1 & -235 & 19023 & -636385 & 8245276 & -29400480 & 0 & 0 \\ 0 & 0 & 6 & -1230 & 83538 & -2206770 & 20268456 & -39916800 & 0 \\ 0 & 0 & 1 & -235 & 19023 & -636385 & 8245276 & -29400480 & 0 \\ 0 & 0 & 0 & 6 & -1230 & 83538 & -2206770 & 20268456 & -39916800 \\ 0 & 0 & 0 & 1 & -235 & 19023 & -636385 & 8245276 & -29400480 \\ 0 & 0 & 0 & 0 & 6 & -1230 & 83538 & -2206770 & 20268456 \end{bmatrix}$$

$$\begin{bmatrix} a(11,10) & a(11,8) & a(11,6) & a(11,4) & a(11,2) & a(11,0) & 0 & 0 & 0 \\ a(11,11) & a(11,9) & a(11,7) & a(11,5) & a(11,3) & a(11,1) & 0 & 0 & 0 \\ 0 & a(11,10) & a(11,8) & a(11,6) & a(11,4) & a(11,2) & a(11,0) & 0 & 0 \\ 0 & a(11,11) & a(11,9) & a(11,7) & a(11,5) & a(11,3) & a(11,1) & 0 & 0 \\ 0 & 0 & a(11,10) & a(11,8) & a(11,6) & a(11,4) & a(11,2) & a(11,0) & 0 \\ 0 & 0 & a(11,11) & a(11,9) & a(11,7) & a(11,5) & a(11,3) & a(11,1) & 0 \\ 0 & 0 & 0 & a(11,10) & a(11,8) & a(11,6) & a(11,4) & a(11,2) & a(11,0) \\ 0 & 0 & 0 & a(11,11) & a(11,9) & a(11,7) & a(11,5) & a(11,3) & a(11,1) \\ 0 & 0 & 0 & 0 & a(11,10) & a(11,8) & a(11,6) & a(11,4) & a(11,2) \end{bmatrix}$$

$$\begin{aligned} \det(M_{11,0}) &= 1027755926256566082207744000000000 \\ \det(M_{11,1}) &= 2832219814419549388880000000 \\ \det(M_{11,2}) &= 780483855384576000000 \\ \det(M_{11,3}) &= -4301608550400000 \\ \det(M_{11,4}) &= 213373440000 \\ \det(M_{11,5}) &= 31752000 \\ \det(M_{11,6}) &= 37800 \\ \det(M_{11,7}) &= -180 \\ \det(M_{11,8}) &= 6. \end{aligned}$$

Based on enormous amount of computations, we eventually realize that this technique works well for the Odd graph O_9 simply because of its intersection numbers satisfy the conditions $a_0 = a_1 = \cdots = a_7 = 0$ and $a_8 \neq 0$, rather than the eigenvalues themselves. This observation also explains why it may not work for bipartite case in general because $a_0 = a_1 = a_2 = \cdots = a_{d-1} = a_d = 0$ where d is its diameter. Indeed, the procedure used for O_9 has been modified successfully so that it can be applied to any connected regular graph with the same Hoffman polynomial as that of a generalized Odd graph. We now state it formally in the following.

Theorem 4.1 *A connected regular graph with the same set of distinct eigenvalues and with the same umber of vertices as those of a generalized Odd graph must be distance-regular with the identical intersection array.*

As a consequence, following a known parametric characterization of O_d [4, p.260], we confirm a question asked by Cvetković [6, p.36] that whether Odd graphs can be characterized by their spectra among connected regular graphs.

Corollary 4.2 *If Γ is a connected regular graph with $Spec(\Gamma) = Spec(O_d)$ for some d, then Γ is isomorphic to O_d.*

The ideas behind the proof are summarized in the following. Let Γ be a connected regular graph with an adjacency matrix A and with $Spec(\Gamma) = Spec(G) = (k^1, \theta_1^{m_1}, \theta_2^{m_2}, \cdots, \theta_d^{m_d})$ where G is a generalized Odd graph with intersection array $\{b_0, b_1, b_2, \cdots, b_{d-1}; c_1, c_2, c_3, \cdots, c_d\}$. Since the graph in question is a generalized Odd graph, it suffices to show that the (x, y)-entry of A^i depends only on the distance i between them.

Procedures:

1) Let $q(x) = \prod_{i=1}^{d}(x-\theta_i) = \sum_{i=0}^{d} q_i x^i$, then $q_d = 1$, $q_{d-1} = c_d$, $q_{d-2} = c_{d-1}c_d - (b_0c_1 + \cdots + b_{d-2}c_{d-1})$, and other coefficients q_{d-i} can be expressed systematically in terms of the intersection numbers.

2) Since $\sum_{j=0}^{d} m_j\theta_j^{2j+1} = 0$, $A^{2i+1}(x,x) = 0$ for $x \in V(\Gamma)$ and $i \leq d-1$. Moreover, $A^{2i+1-j}(x,y) = 0$ if $\partial(x,y) = j$, and hence $a_i(x,y) = 0$ if $\partial(x,y) = i \leq d-1$, ref to lemma 2.1.

3) Multiplying A^i on both sides of the equation $q(A) = vJ$, where $v = \frac{q(k)}{1+\sum_{i=1}^{d} m_i}$, then $A^i q(A) = \theta_0^i \cdot v \cdot J$ for $i \leq d-1$.

4) Combining 1) and 2), translate the information in 3) into a set of systems of linear equations in variables $A^i(x,y)$ as given below

$$\left[\begin{array}{ccccccccccccc} q_{d-1} & q_{d-3} & q_{d-5} & \cdots & q_5 & q_3 & q_1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 1 & q_{d-2} & q_{d-4} & \cdots & q_6 & q_4 & q_2 & q_0 & 0 & 0 & \cdots & 0 & 0 \\ \hline 0 & q_{d-1} & q_{d-3} & \cdots & q_7 & q_5 & q_3 & q_1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & q_{d-2} & \cdots & q_8 & q_6 & q_4 & q_2 & q_0 & 0 & \cdots & 0 & 0 \\ \hline \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \hline 0 & 0 & 0 & \cdots & 0 & q_{d-1} & q_{d-3} & q_{d-5} & q_{d-7} & q_{d-9} & \cdots & q_3 & q_1 \\ 0 & 0 & 0 & \cdots & 0 & 1 & q_{d-2} & q_{d-4} & q_{d-6} & q_{d-8} & \cdots & q_4 & q_2 \end{array}\right] \left[\begin{array}{c} A^{2d-4}(x,y) \\ A^{2d-6}(x,y) \\ A^{2d-8}(x,y) \\ A^{2d-10}(x,y) \\ \vdots \\ A^4(x,y) \\ A^2(x,y) \end{array}\right]$$

$$= v\left[\begin{array}{ccccccc} k^{d-3} & k^{d-4} & k^{d-5} & k^{d-6} & \cdots & k^1 & k^0 \end{array}\right]^t$$

in case d is even, and it is similar if d is odd.

5) Deriving relationship among some auxiliary expressions in terms of $\{b_0, b_1, b_2, \cdots, b_{d-1}; c_1, c_2, c_3, \cdots, c_d\}$, so that a series of row operations can be performed over these coefficient matrices in an algorithmic way to claim their nonsingularity, and hence $A^i(x,y)$'s are constants for suitable i and $\partial(x,y)$, but independent of the choices of x and y in $V(\Gamma)$.

6) Up to this point, concluding the distance-regularity of Γ with identical intersection array $\{b_0, b_1, b_2, \cdots, b_{d-1}; c_1, c_2, c_3, \cdots, c_d\}$. Γ is isomorphic to G if parametric characterization of G is available.

The detailed proof is now under preparation, and will be published soon. We expect that the idea and these procedures will also be available for some other family of graphs with high degree of regularity if step 2 is modified appropriately.

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