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THREE-DIMENSIONAL CELLULAR NEURAL NETWORKS AND PATTERN GENERATION PROBLEMS

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This work investigates three-dimensional pattern generation problems and their applications to three-dimensional Cellular Neural Networks (3DCNN). An ordering matrix for the set of all local patterns is established to derive a recursive formula for the ordering matrix of a larger finite lattice. For a given admissible set of local patterns, the transition matrix is defined and the recursive formula of high order transition matrix is presented. Then, the spatial entropy is obtained by computing the maximum eigenvalues of a sequence of transition matrices. The connecting operators are used to verify the positivity of the spatial entropy, which is important in determining the complexity of the set of admissible global patterns. The results are useful in studying a set of global stationary solutions in various Lattice Dynamical Systems and Cellular Neural Networks.

Keywords: Three-dimensional Cellular Neural Networks; Lattice Dynamical Systems; spatial entropy; pattern generation; connecting operator.

1. Introduction

Lattice Dynamical Systems arise naturally in a wide range of applications of scientific models, in such fields as phase transitions [Baxter, 1971; Cahn, 1960; Lieb, 1967a, 1967b, 1967c, 1967d, 1970; Onsager, 1944] and [Yang & Yang, 1966a, 1966b, 1966c], biology [Bell, 1981; Bell & Cosner, 1984; Ermentrout, 1992; Ermentrout & Kopell, 1994; Ermentrout *et al.*, 1991; Keener, 1987, 1991] and [Kimball *et al.*, 1993], chemical reaction [Bates & Chmaj, 1999; Bates *et al.*, 2001] and [Eveneux & Laplante, 1992], image processing and pattern recognition [Chua, 1998; Chua *et al.*, 1995; Chua & Roska, 1993; Chua & Yang, 1988a, 1988b; Firth, 1988] and

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[Itoh & Chua, 2003, 2005]. In recent years, much attention has been paid to the complexity of the set of all global patterns, with particular reference to spatial entropy [Ban *et al.*, 2001a; Ban *et al.*, 2002; Ban *et al.*, 2001b; Ban & Lin, 2005; Ban *et al.*, 2007, 2008a, 2008b; Chow & Mallet-Paret, 1995; Chow *et al.*, 1996a, 1996b; Hsu *et al.*, 2000; Juang & Lin, 2000; Juang *et al.*, 2000; Lin & Yang, 2000, 2002; Lind & Marcus, 1995] and [Markley & Paul, 1979, 1981].

In the one-dimensional case, spatial entropy h can be determined exactly using an associated transition matrix $\mathbb{T}$, i.e. $h = \log\lambda(\mathbb{T})$, where $\lambda(\mathbb{T})$ is the maximum eigenvalue of $\mathbb{T}$.

For a two-dimensional situation, Ban and Lin [2005] developed a systematical approach for determining the high order transition matrix $\mathbb{T}_n$. They determined the spatial entropy h by computing the maximum eigenvalues of a sequence of such transition matrices. For a class of admissible local patterns, meaning for a class of $\mathbb{T}_2$, the limiting equation of $\rho^* = \exp(h(\mathbb{T}_2))$ can be exactly solved using the recursive formula for $\rho(\mathbb{T}_n)$. However, $\mathbb{T}_n$ is a $2^n \times 2^n$ matrix, and $\rho(\mathbb{T}_n)$ is usually quite difficult to compute for large n. The connecting operator and the trace operator have been derived to overcome these difficulties [Ban *et al.*, 2007]; lower-bound estimates of entropy have been obtained by introducing connecting operators $\mathbb{C}_m$, and upper-bound estimates of entropy have been made by introducing trace operators $\mathcal{T}_m$.

This work develops a general method to investigate three-dimensional pattern generation problems, extending other studies [Ban & Lin, 2005] and [Ban *et al.*, 2007] to the three-dimensional case. It focuses on ordering matrices of patterns and on the connecting operator in the three-dimensional case. The trace operator has been described elsewhere [Ban *et al.*, 2008b]. This work is motivated by 3DCNN, so it is a major tool to study global patterns in 3DCNN.

Three-dimensional pattern generation problems are considered initially. Let $\mathcal{S}$ be a finite set of $p \geq 2$ colors, where $\mathbf{Z}^3$ denotes the integer lattice of $\mathbb{R}^3$. Denote, $U : \mathbf{Z}^3 \to \mathcal{S}$, a global pattern by $U(\alpha_1, \alpha_2, \alpha_3) = u_{\alpha_1\alpha_2\alpha_3}$. The set of all patterns with p colors in a three-dimensional lattice is expressed as $\Sigma_p^3 \equiv \mathcal{S}^{\mathbf{Z}^3} = \{U | U : \mathbf{Z}^3 \to \mathcal{S}\}$. The set of all local patterns on $\mathbf{Z}_{m_1\times m_2\times m_3}$ is denoted by

$$\Sigma_{m_1\times m_2\times m_3} \equiv \{U|_{\mathbf{Z}_{m_1\times m_2\times m_3}} | U \in \Sigma_p^3\}$$

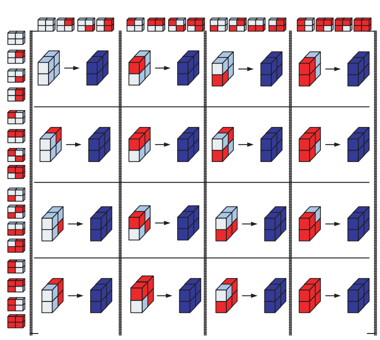
where $\mathbf{Z}_{m_1\times m_2\times m_3} = \{(\alpha_1, \alpha_2, \alpha_3)|\ 1 \leq \alpha_i \leq m_i,\ 1 \leq i \leq 3\}$ is an $m_1 \times m_2 \times m_3$ finite rectangular lattice. For simplicity, two colors on the $2 \times 2 \times 2$ lattice $\mathbf{Z}_{2\times2\times2}$ are considered here. Given a basic set $\mathcal{B} \subset \Sigma_{2\times2\times2}$, the spatial entropy can be defined as

$$h(\mathcal{B}) = \lim_{m_1,m_2,m_3\to\infty} \frac{\log \Gamma_{m_1\times m_2\times m_3}(\mathcal{B})}{m_1 m_2 m_3}, \qquad (1)$$

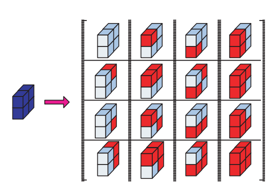
where $\Gamma_{m_1\times m_2\times m_3}(\mathcal{B})$ is the number of distinct patterns in $\Sigma_{m_1\times m_2\times m_3}(\mathcal{B})$ and $\Sigma_{m_1\times m_2\times m_3}(\mathcal{B})$ is the set of all local patterns on $\mathbf{Z}_{m_1\times m_2\times m_3}$, which can be generated from $\mathcal{B}$, as described elsewhere [Chow *et al.*, 1996b]. Six different orderings

$$\begin{aligned}
[x] &: [1] \succ [2] \succ [3]\\
[y] &: [2] \succ [1] \succ [3]\\
[z] &: [3] \succ [1] \succ [2]\\
[\hat{x}] &: [1] \succ [3] \succ [2]\\
[\hat{y}] &: [2] \succ [3] \succ [1]\\
[\hat{z}] &: [3] \succ [2] \succ [1]
\end{aligned}$$

are obtained and the ordering matrix $\mathbb{W}_{2\times2\times2}$ for $\Sigma_{2\times2\times2}$ can be introduced according to the different ordering $[\omega]$. Without loss of generality, $\mathbb{X}_{2\times2\times2}$ is considered



where



(2)

and the other cases are similar.

One of the main results is the construction of $\hat{\mathbb{X}}_{2\times m_2\times m_3}$ from $\mathbb{X}_{2\times2\times2}$, where $\hat{\mathbb{X}}_{2\times m_2\times m_3}$ represents the ordering matrix of $\Sigma_{2\times m_2\times m_3}$ according to $[\hat{x}]$-ordering. It can be addressed in the following three steps.

Step I. Apply $[x]$-ordering to $\mathbf{Z}_{1\times m_2\times 2}$

2	4	...	2k	...	2m$_2$-2	2m$_2$
1	3	...	2k-1	...	2m$_2$-3	2m$_2$-1

y →

and introduce ordering matrix $\mathbb{X}_{2\times m_2\times 2}$ for $\Sigma_{2\times m_2\times 2}$ as in Theorem 2.1. By Theorem 3.2, the transition matrix $\mathbb{A}_{x;2\times m_2\times 2}$ can be obtained from

$$\mathbb{A}_{x;2\times m_2\times 2} = (\mathbb{A}_{x;2\times(m_2-1)\times 2})_{2^{2(m_2-1)}\times 2^{2(m_2-1)}} \circ (E_{2^{2(m_2-2)}} \otimes \mathbb{A}_{x;2\times 2\times 2}),$$

where E_{2^k} is the $2^k \times 2^k$ matrix with 1 as its entries, $\otimes$ is the tensor product and $\circ$ is the Hadamard product, as in Eq. (36).

Step II. Convert $[x]$-ordering into $[\hat{x}]$-ordering on $\mathbf{Z}_{1\times m_2\times 2}$ using

z ↑

m$_2$+1	m$_2$+2	...	m$_2$+k	...	2m$_2$
1	2	...	k	...	m$_2$

and introduce the ordering matrix $\hat{\mathbb{X}}_{2\times m_2\times 2}$ for $\Sigma_{2\times m_2\times 2}$ as in Theorem 2.4. The associated transition matrix $\mathbb{A}_{\hat{x};2\times m_2\times 2}$ is given by

$$\mathbb{A}_{\hat{x};2\times m_2\times 2} = \mathbb{P}^t_{x;2\times m_2\times 2}\mathbb{A}_{x;2\times m_2\times 2}\mathbb{P}_{x;2\times m_2\times 2},$$

where $\mathbb{P}_{x;2\times m_2\times 2}$ is the permutation matrix as in Theorem 3.4.

Step III. Define $[\hat{x}]$-ordering on $\mathbf{Z}_{1\times m_2\times m_3}$ as

z ↑

(m$_3$-1)m$_2$+1	(m$_3$-1)m$_2$+2	...	m$_3$m$_2$-1	m$_3$m$_2$
:	:	:	:	:
m$_2$+1	m$_2$+2	...	2m$_2$-1	2m$_2$
1	2	...	m$_2$-1	m$_2$

and introduce ordering matrix $\hat{\mathbb{X}}_{2\times m_2\times m_3}$ for $\Sigma_{2\times m_2\times m_3}$ as in Theorem 2.5. The recursive formula for the transition matrix $\mathbb{A}_{\hat{x};2\times m_2\times m_3}$ can be obtained by

$$\mathbb{A}_{\hat{x};2\times m_2\times m_3} = (\mathbb{A}_{\hat{x};2\times m_2\times(m_3-1)})_{2^{m_2(m_3-1)}\times 2^{m_2(m_3-1)}} \circ (E_{2^{m_2(m_3-2)}} \otimes \mathbb{A}_{\hat{x};2\times m_2\times 2})$$

as in Theorem 3.5.

Theorem 3.7 enables the maximum eigenvalue $\lambda_{\hat{x};2,m_2,m_3}$ of $\mathbb{A}_{\hat{x};2\times m_2\times m_3}$ to be computed, to yield the spatial entropy,

$$h(\mathcal{B}) = \lim_{m_2,m_3\to\infty} \frac{\log \lambda_{\hat{x};2,m_2,m_3}}{m_2 m_3}.$$

However, some estimates of lower bound of spatial entropy $h(\mathcal{B})$ can be made using the connecting operator. Then, for fixed m_1, $m_2 \geq 2$, the m_3-limit in Eq. (1) is studied:

$$\lim_{m_3\to\infty} \frac{1}{m_3} \log |\mathbb{A}^{m_1}_{\hat{x};2\times m_2\times m_3}|. \tag{3}$$

The recursive formula of $\mathbb{A}^{m_1}_{\hat{x};2\times m_2\times m_3}$ in m_3 is considered. Accordingly, the next task is to investigate Eq. (3). According to Eqs. (53) and (54),

$$\mathbb{A}^{m_1}_{\hat{x};2\times m_2\times m_3} = [A_{\hat{x};m_1,m_2,m_3;\alpha}]_{2^{m_2}\times 2^{m_2}},$$

$$A_{\hat{x};m_1,m_2,m_3;\alpha} = \sum_{k=1}^{2^{m_2(m_1-1)}} A^{(k)}_{\hat{x};m_1,m_2,m_3;\alpha}$$

where $A^{(k)}_{\hat{x};m_1,m_2,m_3;\alpha}$ is called an elementary pattern of order (m_1, m_2, m_3) and is a fundamental element in constructing $A_{\hat{x};m_1,m_2,m_3;\alpha}$. $\mathbb{V}_{\hat{x};m_1,m_2,m_3}$ is defined as

$$\mathbb{V}_{\hat{x};m_1,m_2,m_3} = [V_{\hat{x};m_1,m_2,m_3;\alpha}],$$
$$V_{\hat{x};m_1,m_2,m_3;\alpha} = (A^{(k)}_{\hat{x};m_1,m_2,m_3;\alpha})^t$$

as in Eqs. (55) and (56), which specifies systematically these elementary patterns. The connecting operator $\mathbb{C}_{\hat{x};m_3;m_1m_2}$ is introduced as in Definition 4.2, and used to derive a recursive formula for $A^{(k)}_{\hat{x};m_1,m_2,(m_3+1);\alpha_1;\alpha_2}$ and $A^{(\ell)}_{\hat{x};m_1,m_2,m_3;\alpha_2}$ as in Theorem 4.5

$$V_{\hat{x};m_1,m_2,m_3+1;\alpha_1;\alpha_2} = S_{\hat{x};m_3;m_1m_2;\alpha_1\alpha_2} V_{\hat{x};m_1,m_2,m_3;\alpha_2},$$

where $\mathbb{C}_{\hat{x};m_3;m_1m_2} = \mathbb{S}^{(r)}_{\hat{x};m_3;m_1m_2}$. The recursive formula Eq. (67) yields a lower bound on entropy

$$h(\mathbb{A}_{x;2\times 2\times 2}) \geq \lim_{m_2\to\infty} \frac{1}{m_1 m_2 P} \log \rho(S_{\hat{x};m_3;m_1m_2;\alpha_1\alpha_2} S_{\hat{x};m_3;m_1m_2;\alpha_2\alpha_3} \cdots S_{\hat{x};m_3;m_1m_2;\alpha_P\alpha_1}) \tag{4}$$

such as in Theorem 4.12 and which implies $h(\mathbb{A}_{x;2\times2\times2}) > 0$ if a diagonal periodic cycle is applied with a limit in Eq. (4) that exceeds 0. This method powerfully yields the positivity of spatial entropy, which is useful in evaluating the complexity of patterns generation problems.

The method is very effective in elucidating the complexity of the set of mosaic patterns in 3DCNN. A typical 3DCNN is of the form

$$\frac{du_{i,j,k}}{dt} = -u_{i,j,k} + w + \sum_{|\alpha|,|\beta|,|\gamma|\le 1} a_{\alpha,\beta,\gamma} f(u_{i+\alpha,j+\beta,k+\gamma}) + \sum_{|\alpha|,|\beta|,|\gamma|\le 1} b_{\alpha,\beta,\gamma} u_{i+\alpha,j+\beta,k+\gamma}, \tag{5}$$

where $(i,j,k) \in \mathbf{Z}^3$, $f(u)$ is a piecewise-linear output function, defined by

$$v = f(u) = \frac{1}{2}(|u+1| - |u-1|).$$

Here, $A = (a_{\alpha,\beta,\gamma})$ is a feedback template, a spatial-invariant template; $B = (b_{\alpha,\beta,\gamma})$ is a controlling template, and w is called a biased term or threshold. To elucidate the method, consider nonzero $a_{0,0,0} = a$, $a_{1,0,0} = a_x$, $a_{0,1,0} = a_y$, $a_{0,0,1} = a_z$ and zero other $a_{\alpha,\beta,\gamma}$ and $b_{\alpha,\beta,\gamma}$. Therefore, Eq. (5) can be rewritten as

$$\frac{du_{i,j,k}}{dt} = -u_{i,j,k} + w + af(u_{i,j,k}) + a_x f(u_{i+1,j,k}) + a_y f(u_{i,j+1,k}) + a_z f(u_{i,j,k+1}). \tag{6}$$

The quantities $u_{i,j,k}$ represent the state of cell at (i,j,k). The stationary solution $\bar{u} = (\bar{u}_{i,j,k})$ of Eq. (6) satisfies

$$u_{i,j,k} = w + av_{i,j,k} + a_x v_{i+1,j,k} + a_y v_{i,j+1,k} + a_z v_{i,j,k+1}, \tag{7}$$

where $v = f(u)$, which is very important in studying 3DCNNs: their outputs $\bar{v} = (\bar{v}_{i,j,k}) = f(\bar{u}_{i,j,k})$ are called patterns. A mosaic solution $\bar{u}$ satisfies $|\bar{u}_{i,j,k}| \ge 1$ and its corresponding pattern $\bar{v}$ is called a mosaic pattern here $|\bar{u}_{i,j,k}| \ge 1$ for all $(i,j,k) \in \mathbf{Z}^3$. Among the stationary solutions, the mosaic solutions are stable and are crucial to study the complexity of Eq. (6). Equation (7) has five parameters w, a, a_x, a_y and a_z. $a_x < a_y < a_z < 0$ and $|a_x| > |a_y| + |a_z|$ are considered to elucidate application of our work. In particular, region [4, 8] in Fig. 4 in Sec. 5 is considered: the transition matrix can be written as

$$\mathbb{A}_{x;2\times2\times2} = G \otimes E \otimes E \otimes E,$$

where $G = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ and $E = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$.

Then, Steps I–III yield the aforementioned admissible patterns in $\Sigma_{2\times m_2\times m_3}$; the corresponding transition matrix can be derived as in Proposition 3.9.

Step I $\Rightarrow \mathbb{A}_{x;2\times m_2\times 2} = \otimes(G\otimes E)^{m_2-1} \otimes (\otimes E^2)$,
Step II $\Rightarrow \mathbb{A}_{\hat{x};2\times m_2\times 2} = (\otimes G^{m_2-1}) \otimes (\otimes E^{m_2+1})$,
Step III $\Rightarrow \mathbb{A}_{\hat{x};2\times m_2\times m_3} = \otimes((\otimes G^{m_2-1})\otimes E)^{m_3-1} \otimes (\otimes E^{m_2})$.

The complexity of the 3DCNN model, as in Eq. (6), can be examined using the connecting operator defined in Sec. 4. Since the connecting operator

$$C_{z;m_1;m_2 2;11} = S_{z;m_1;m_2 2;11} = (\otimes G^{m_2-1}) \otimes E,$$

the maximum eigenvalue can be exactly computed as

$$\rho(S_{z;m_1;m_2 2;11}) = 2g^{m_2-1},$$

where $g = (1+\sqrt{5})/2$ is the golden-mean, as in Proposition 5.1. According to Eq. (4), the lower bound of spatial entropy in the region (VIII)-(i)-(1)-[4,8] can be estimated

$$h(\mathbb{A}_{x;2\times2\times2}) \ge \lim_{m_2\to\infty} \frac{1}{2m_2} \log \rho(S_{z;m_1;m_2 2;11}) = \frac{1}{2}\log g.$$

Moreover, in this case, spatial entropy can be solved exactly from the maximum eigenvalue of $\mathbb{A}_{\hat{x};2\times m_2\times m_3}$. Since

$$\rho(\mathbb{A}_{\hat{x};2\times m_2\times m_3}) = 2^{m_2+m_3-1} g^{(m_2-1)(m_3-1)},$$

the spatial entropy is

$$h(\mathbb{A}_{x;2\times2\times2}) = \lim_{m_2,m_3\to\infty} \frac{\rho(\mathbb{A}_{\hat{x};2\times m_2\times m_3})}{m_2 m_3} = \log g$$

as in Proposition 3.9.

The rest of this paper is organized as follows. Section 2 derives a recursive formula for the ordering matrix $\mathbb{X}_{2\times m_2\times 2}$ for $\Sigma_{2\times m_2\times 2}$ from $\mathbb{X}_{2\times2\times2}$. The ordering $[x]$ is converted to $[\hat{x}]$. Then, a similar recursive formula is constructed for ordering matrix $\hat{\mathbb{X}}_{2\times m_2\times m_3}$ from $\hat{\mathbb{X}}_{2\times m_2\times 2}$. Section 3 derives the recursive formula for the associated high order transition matrices $\mathbb{A}_{\hat{x};2\times m_2\times m_3}$ from $\mathbb{A}_{x;2\times2\times2}$. Section 4 derives the connecting operator $\mathbb{C}_{\hat{x};m_3;m_1m_2}$, which can recursively reduce elementary patterns of high order to patterns of low order. Then, the lower-bound of spatial entropy is determined by computing the maximum eigenvalues of the diagonal periodic cycles of sequence $S_{\hat{x};m_3;m_1m_2;\alpha\beta}$. Section 5 gives an example of the application of our main results to 3DCNN.

2. Three-Dimensional Pattern Generation Problems

This section describes three-dimensional pattern generation problems. Here, m_1, m_2, $m_3 \geq 2$ are fixed and indices are omitted for brevity. Let $\mathcal{S}$ be a set of p colors, and $\mathbf{Z}_{m_1\times m_2\times m_3}$ be a fixed finite rectangular sublattice of $\mathbf{Z}^3$, where $\mathbf{Z}^3$ denotes the integer lattice on $\mathbb{R}^3$ and (m_1, m_2, m_3) a three-tuple of positive integers. Functions $U : \mathbf{Z}^3 \to \mathcal{S}$ and $U_{m_1\times m_2\times m_3} : \mathbf{Z}_{m_1\times m_2\times m_3} \to \mathcal{S}$ are called global patterns and local patterns on $\mathbf{Z}_{m_1\times m_2\times m_3}$ respectively. The set of all patterns U is denoted by $\Sigma_p^3 \equiv \mathcal{S}^{\mathbf{Z}^3}$, such that Σ_p^3 is the set of all patterns with p different colors in a three-dimensional lattice. For clarity, two symbols, $\mathcal{S} = \{0, 1\}$ are considered. Let x, y and z coordinate represent the 1st-, 2nd- and 3rd-coordinates respectively as in Fig. 1.

Six orderings $[\omega]$ ordering are represented as Eq. (8)

$$\begin{aligned}
&[x] : [1] \succ [2] \succ [3]\\
&[y] : [2] \succ [1] \succ [3]\\
&[z] : [3] \succ [1] \succ [2]\\
&[\hat{x}] : [1] \succ [3] \succ [2]\\
&[\hat{y}] : [2] \succ [3] \succ [1]\\
&[\hat{z}] : [3] \succ [2] \succ [1]
\end{aligned} \tag{8}$$

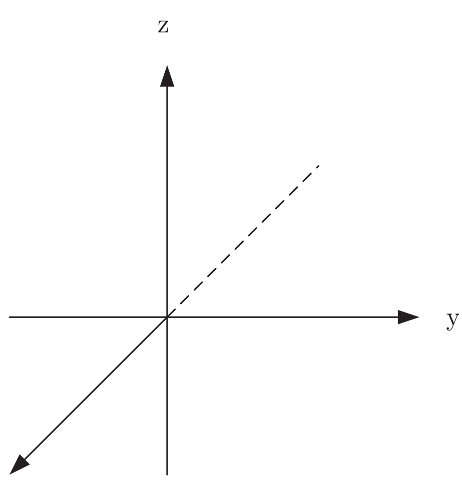


Fig. 1. Three-dimensional coordinate system.

On a fixed finite lattice $\mathbf{Z}_{m_1\times m_2\times m_3}$, an ordering $[\omega] : [i] \succ [j] \succ [k]$ is obtained on $\mathbf{Z}_{m_1\times m_2\times m_3}$, which is any one of the above orderings on $\mathbf{Z}_{m_1\times m_2\times m_3}$

$$\psi_\omega(\alpha_1, \alpha_2, \alpha_3) = m_j m_k(\alpha_i - 1) + m_k(\alpha_j - 1) + \alpha_k,$$

where $1 \leq \alpha_\ell \leq m_\ell$ and $1 \leq \ell \leq 3$. The ordering $[\omega]$ on $\mathbf{Z}_{m_1\times m_2\times m_3}$ can now be applied to $\Sigma_{m_1\times m_2\times m_3}$. Indeed, for each $U = (u_{\alpha_1\alpha_2\alpha_3}{}^1) \in \Sigma_{m_1\times m_2\times m_3}$, define

$$\begin{aligned}
\psi_\omega(U) &\equiv \psi_{\omega;m_1,m_2,m_3}(U)\\
&\equiv 1 + \sum_{\alpha_i=1}^{m_i}\sum_{\alpha_j=1}^{m_j}\sum_{\alpha_k=1}^{m_k} u_{\alpha_1\alpha_2\alpha_3}\omega_{m_i,m_j,m_k}^{\alpha_i,\alpha_j,\alpha_k},
\end{aligned}$$

where

$$\begin{aligned}
\omega_{m_i,m_j,m_k}^{\alpha_i,\alpha_j,\alpha_k} &= 2^{m_i m_j m_k - \psi_\omega(\alpha_1,\alpha_2,\alpha_3)}\\
&= 2^{m_k m_j(m_i-\alpha_i)+m_k(m_j-\alpha_j)+(m_k-\alpha_k)}.
\end{aligned}$$

U is referred to herein as the $\psi_\omega(U)$-th element in $\Sigma_{m_1\times m_2\times m_3}$ by ordering $[\omega]$. Identifying the pictorial patterns using $\psi_\omega(U)$ is very effective in proving theorems since computations can now be performed on $\psi_\omega(U)$. For instance, the orderings on $\mathbf{Z}_{2\times 2\times 2}$ can be represented as in Fig. 2.

2.1. *Ordering matrices*

The cube $\mathbf{Z}_{m_1\times m_2\times m_3}$ can be decomposed by m_1-many (m_2-many and m_3-many) parallel two-dimensional rectangles in $\mathbf{Z}_{1\times m_2\times m_3}$ ($\mathbf{Z}_{m_1\times 1\times m_3}$ and $\mathbf{Z}_{m_1\times m_2\times 1}$). Any patterns $U = (u_{\alpha_1\alpha_2\alpha_3}) \in \Sigma_{m_1\times m_2\times m_3}$ can be decomposed accordingly. For example, in $[x]$-ordering, define the α_1th layer of rectangle as

$$\begin{aligned}
\mathbf{Z}_{\alpha_1;m_2\times m_3} = \{(\alpha_1, \alpha_2, \alpha_3) | 1 \leq \alpha_2 \leq m_2,\\
1 \leq \alpha_3 \leq m_3\}.
\end{aligned}$$

Pattern U in α_1th layer is assigned the number

$$i_{\alpha_1} \equiv 1 + \sum_{\alpha_2=1}^{m_2}\sum_{\alpha_3=1}^{m_3} u_{\alpha_1\alpha_2\alpha_3} x_{1,m_2,m_3}^{1,\alpha_2,\alpha_3}, \tag{9}$$

where $x_{1,m_2,m_3}^{1,\alpha_2,\alpha_3} = 2^{m_2m_3 - m_3(\alpha_2-1)-\alpha_3}$. As denoted by the $1 \times m_2 \times m_3$ pattern

$x_{1\times m_2\times m_3;i_{\alpha_1}} =$

$u_{\alpha_1 1 m_3}$	$u_{\alpha_1 2 m_3}$	$\cdots$	$u_{\alpha_1 m_2 m_3}$
$\vdots$	$\vdots$	$\ddots$	$\vdots$
$u_{\alpha_1 12}$	$u_{\alpha_1 22}$	$\cdots$	$u_{\alpha_1 m_2 2}$
$u_{\alpha_1 11}$	$u_{\alpha_1 21}$	$\cdots$	$u_{\alpha_1 m_2 1}$

.

[1]Use $u_{\alpha_1\alpha_2\alpha_3}$ to substitute $u_{\alpha_1,\alpha_2,\alpha_3}$ for simplicity afterward.

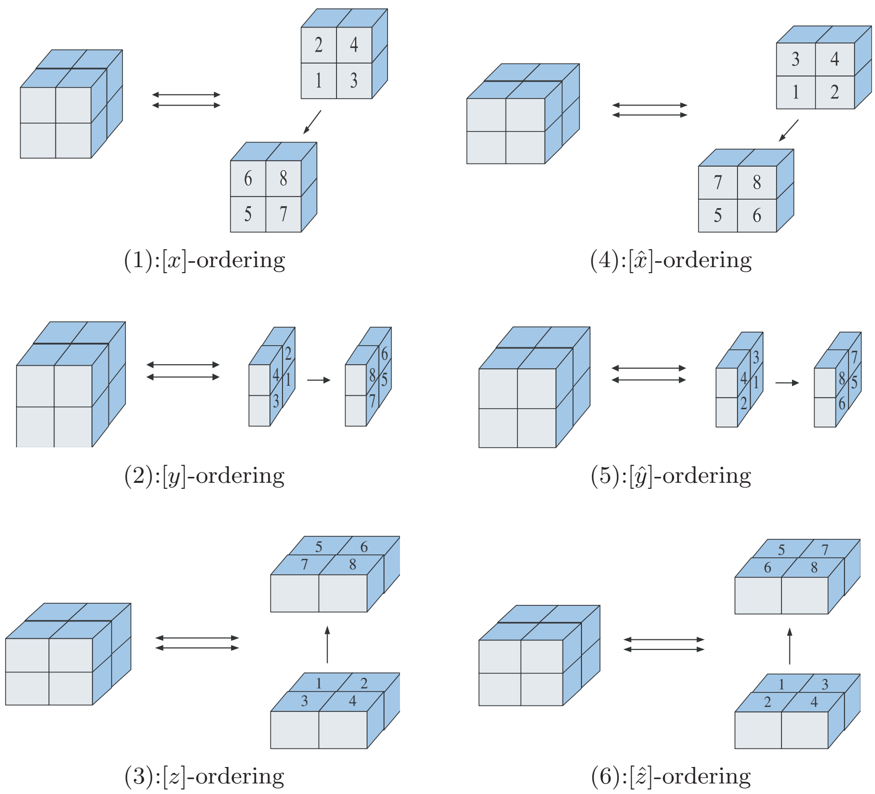


Fig. 2. The orderings of $\mathbf{Z}_{2\times2\times2}$.

In particular, when $m_2 = 2$ and $m_3 = 2$, as denoted by $x_{1\times2\times2;i_{\alpha_1}}$, where

$$i_{\alpha_1} = 1 + 2^3 u_{\alpha_1 11} + 2^2 u_{\alpha_1 12} + 2u_{\alpha_1 21} + u_{\alpha_1 22} \tag{10}$$

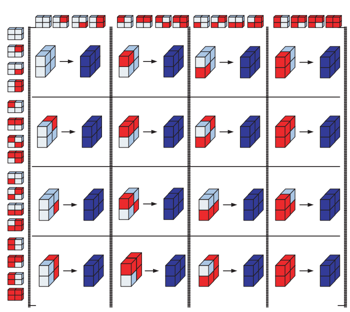
and

$$x_{1\times2\times2;i_{\alpha_1}} \equiv x_{i_{\alpha_1}} = \begin{array}{|c|c|}\hline u_{\alpha_1 12} & u_{\alpha_1 22} \\ \hline u_{\alpha_1 11} & u_{\alpha_1 21} \\ \hline \end{array},$$

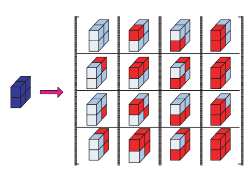
where $\alpha_1 \in \{1,2\}$. A $2\times2\times2$ pattern $U = (u_{\alpha_1\alpha_2\alpha_3})$ can now be obtained from the [x]-direct sum of two $1\times2\times2$ patterns using [x]-ordering:

$$\begin{aligned} x_{2\times2\times2;i_1 i_2} &\equiv x_{i_1 i_2} \\ &\equiv x_{i_1} \oplus x_{i_2} \\ &= \begin{array}{cc} & \begin{array}{|c|c|}\hline u_{112} & u_{122} \\ \hline u_{111} & u_{121} \\ \hline \end{array} \\ & \swarrow \\ \begin{array}{|c|c|}\hline u_{212} & u_{222} \\ \hline u_{211} & u_{221} \\ \hline \end{array} & \end{array}, \end{aligned}$$

where i_{α_1} as in Eq. (10) and $\alpha_1 \in \{1,2\}$. Therefore, the complete set of 2^8 patterns in $\Sigma_{2\times2\times2}$ is given by a 16×16 matrix $\mathbb{X}_{2\times2\times2} = [x_{i_1 i_2}{}^2]$ as its entries in



where



(11)

[2]Use $x_{i_1 i_2}$ to substitute x_{i_1,i_2} for simplicity afterward.

That

$$\psi_x(x_{i_1 i_2}) = 2^4(i_1 - 1) + i_2$$

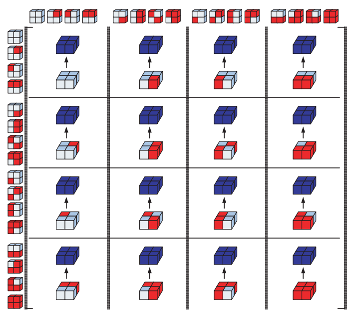
is easily verified, and local patterns in $\Sigma_{2\times2\times2}$ are then counted by going through each row successively in Eq. (11). Correspondingly, $\mathbb{X}_{2\times2\times2}$ can be referred to as an ordering matrix for $\Sigma_{2\times2\times2}$. A $2 \times 2 \times 2$ pattern can also be regarded as an $[x]$-direct sum of two $1 \times 2 \times 2$ patterns using $[\hat{x}]$-ordering,

$$\hat{x}_{2\times2\times2;\hat{i}_1\hat{i}_2} \equiv \hat{x}_{\hat{i}_1\hat{i}_2} \equiv \hat{x}_{\hat{i}_1} \oplus \hat{x}_{\hat{i}_2}$$

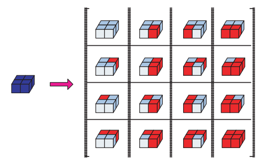
where

$$\hat{i}_{\alpha_1} = 1 + 2^3 u_{\alpha_1 11} + 2^2 u_{\alpha_1 21} + 2u_{\alpha_1 12} + u_{\alpha_1 22}, \quad \alpha_1 \in \{1, 2\}.$$

The ordering matrix $\hat{\mathbb{X}}_{2\times2\times2}$ can be represented as



where

.

Now,

$$\psi_{\hat{x}}(\hat{x}_{\hat{i}_1\hat{i}_2}) = 2^4(\hat{i}_1 - 1) + \hat{i}_2$$

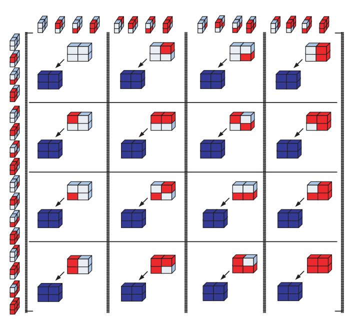
can be verified. Similarly, a $2 \times 2 \times 2$ pattern can also be viewed as a $[y]$-direct ($[\hat{y}]$-direct) and $[z]$-direct ($[\hat{z}]$-direct) sum of $2 \times 1 \times 2$ and $2 \times 2 \times 1$ patterns:

$$y_{j_1 j_2} \equiv y_{j_1} \oplus y_{j_2}, \quad \hat{y}_{\hat{j}_1\hat{j}_2} \equiv \hat{y}_{\hat{j}_1} \oplus \hat{y}_{\hat{j}_2}, \quad z_{k_1 k_2} \equiv z_{k_1} \oplus z_{k_2}, \quad \hat{z}_{\hat{k}_1\hat{k}_2} \equiv \hat{z}_{\hat{k}_1} \oplus \hat{z}_{\hat{k}_2},$$

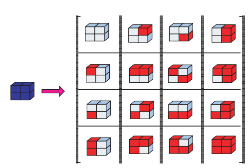
where

$$\begin{aligned} j_{\alpha_2} &= 1 + 2^3 u_{1\alpha_2 1} + 2^2 u_{1\alpha_2 2} + 2u_{2\alpha_2 1} + u_{2\alpha_2 2}, \quad \alpha_2 \in \{1, 2\}, \\ \hat{j}_{\alpha_2} &= 1 + 2^3 u_{1\alpha_2 1} + 2^2 u_{2\alpha_2 1} + 2u_{1\alpha_2 2} + u_{2\alpha_2 2}, \quad \alpha_2 \in \{1, 2\}, \\ k_{\alpha_3} &= 1 + 2^3 u_{11\alpha_3} + 2^2 u_{12\alpha_3} + 2u_{21\alpha_3} + u_{22\alpha_3}, \quad \alpha_3 \in \{1, 2\}, \\ \hat{k}_{\alpha_3} &= 1 + 2^3 u_{11\alpha_3} + 2^2 u_{21\alpha_3} + 2u_{12\alpha_3} + u_{22\alpha_3}, \quad \alpha_3 \in \{1, 2\}. \end{aligned} \tag{12}$$

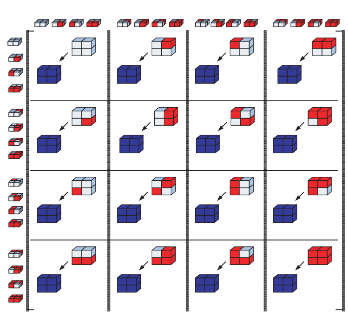
A 16×16 matrix $\mathbb{Y}_{2\times2\times2} = [y_{j_1 j_2}]$ or $\mathbb{Z}_{2\times2\times2} = [z_{k_1 k_2}]$ can also be obtained for $\Sigma_{2\times2\times2}$, such that $\mathbb{Y}_{2\times2\times2} =$



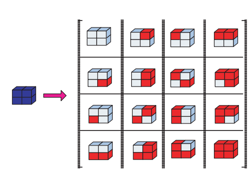
where

,

or $\mathbb{Z}_{2\times2\times2}$



where

.

The relationship between $\mathbb{W}_{2\times 2\times 2}$ must be studied, where $\mathbb{W} \in \{\mathbb{X}, \mathbb{Y}, \mathbb{Z}, \hat{\mathbb{X}}, \hat{\mathbb{Y}}, \hat{\mathbb{Z}}\}$. Before the relations are explained, the column matrix and the row matrix must be given. Let $\mathbb{A} = [a_{ij}]$ be a $m^2 \times m^2$ matrix, the column matrix $\mathbb{A}^{(c)}$ of $\mathbb{A}$ is defined as

$$\mathbb{A}^{(c)} = \begin{bmatrix} A_1^{(c)} & A_2^{(c)} & \cdots & A_m^{(c)} \\ A_{m+1}^{(c)} & A_{m+2}^{(c)} & \cdots & A_{2m}^{(c)} \\ \vdots & \vdots & \ddots & \vdots \\ A_{(m-1)m+1}^{(c)} & A_{(m-1)m+2}^{(c)} & \cdots & A_{m^2}^{(c)} \end{bmatrix},$$

$$A_\alpha^{(c)} = \begin{bmatrix} a_{1\alpha} & a_{2\alpha} & \cdots & a_{m\alpha} \\ a_{(m+1)\alpha} & a_{(m+2)\alpha} & \cdots & a_{(2m)\alpha} \\ \vdots & \vdots & \ddots & \vdots \\ a_{((m-1)m+1)\alpha} & a_{((m-1)m+2)\alpha} & \cdots & a_{m^2\alpha} \end{bmatrix},$$

where $1 \leq \alpha \leq m^2$.

The row matrix $\mathbb{A}^{(r)}$ of $\mathbb{A}$ is defined as

$$\mathbb{A}^{(r)} = \begin{bmatrix} A_1^{(r)} & A_2^{(r)} & \cdots & A_m^{(r)} \\ A_{m+1}^{(r)} & A_{m+2}^{(r)} & \cdots & A_{2m}^{(r)} \\ \vdots & \vdots & \ddots & \vdots \\ A_{(m-1)m+1}^{(r)} & A_{(m-1)m+2}^{(r)} & \cdots & A_{m^2}^{(r)} \end{bmatrix}, \tag{13}$$

$$A_\alpha^{(r)} = \begin{bmatrix} a_{\alpha 1} & a_{\alpha 2} & \cdots & a_{\alpha m} \\ a_{\alpha(m+1)} & a_{\alpha(m+2)} & \cdots & a_{\alpha(2m)} \\ \vdots & \vdots & \ddots & \vdots \\ a_{\alpha((m-1)m+1)} & a_{\alpha((m-1)m+2)} & \cdots & a_{\alpha m^2} \end{bmatrix}, \tag{14}$$

where $1 \leq \alpha \leq m^2$. Hence, based on some observations, $\mathbb{X}_{2\times 2\times 2}$ can be represented in terms of $y_{j_1 j_2}$ as

$$\mathbb{X}_{2\times 2\times 2} = \mathbb{Y}_{2\times 2\times 2}^{(r)}. \tag{15}$$

Furthermore, $\mathbb{Y}_{2\times 2\times 2} = \mathbb{X}_{2\times 2\times 2}^{(r)}$, $\mathbb{Z}_{2\times 2\times 2} = \hat{\mathbb{X}}_{2\times 2\times 2}^{(r)}$, $\hat{\mathbb{X}}_{2\times 2\times 2} = \mathbb{Z}_{2\times 2\times 2}^{(r)}$, $\hat{\mathbb{Y}}_{2\times 2\times 2} = \hat{\mathbb{Z}}_{2\times 2\times 2}^{(r)}$ and $\hat{\mathbb{Z}}_{2\times 2\times 2} = \hat{\mathbb{Y}}_{2\times 2\times 2}^{(r)}$ can also be obtained. The remainder of this subsection addresses the construction of $\hat{\mathbb{X}}_{2\times m_2\times m_3}$ from $\mathbb{X}_{2\times 2\times 2}$ in the following three steps, where $\hat{\mathbb{X}}_{2\times m_2\times m_3}$ represents the ordering matrix of $\Sigma_{2\times m_2\times m_3}$ according to $[\hat{x}]$-ordering generated from $\Sigma_{2\times 2\times 2}$.

Step I. Apply $[x]$-ordering to $\mathbf{Z}_{1\times m_2\times 2}$ using

2	4	...	2k	...	$2m_2$-2	$2m_2$
1	3	...	2k-1	...	$2m_2$-3	$2m_2$-1

(16)

y →

and introduce ordering matrix $\mathbb{X}_{2\times m_2\times 2}$ for $\Sigma_{2\times m_2\times 2}$.

Step II. Convert $[x]$-ordering into $[\hat{x}]$-ordering on $\mathbf{Z}_{1\times m_2\times 2}$ by

m_2+1	m_2+2	...	m_2+k	...	$2m_2$
1	2	...	k	...	m_2

(17)

z ↑

and introduce ordering matrix $\hat{\mathbb{X}}_{2\times m_2\times 2}$ for $\Sigma_{2\times m_2\times 2}$.

Step III. Define $[\hat{x}]$-ordering on $\mathbf{Z}_{1\times m_2\times m_3}$ by

$(m_3$-1$)m_2$+1	$(m_3$-1$)m_2$+2	...	m_3m_2-1	m_3m_2
:	:	:	:	:
m_2+1	m_2+2	...	$2m_2$-1	$2m_2$
1	2	...	m_2-1	m_2

z ↑

(18)

and introduce ordering matrix $\hat{\mathbb{X}}_{2\times m_2\times m_3}$ for $\Sigma_{2\times m_2\times m_3}$.

To introduce $\mathbb{X}_{2\times m_2\times 2}$, define

$$\begin{aligned} y_{2\times m_2\times 2;j_1 j_2\ldots j_{m_2}} &\equiv y_{2\times 2\times 2;j_1 j_2} \hat{\oplus} y_{2\times 2\times 2;j_2 j_3} \\ &\quad \hat{\oplus} \cdots \hat{\oplus} y_{2\times 2\times 2;j_{m_2-1} j_{m_2}} \\ &\equiv y_{j_1} \oplus y_{j_2} \oplus \cdots \oplus y_{j_{m_2}}, \end{aligned} \tag{19}$$

where $1 \leq j_k \leq 2^4$ and $1 \leq k \leq m_2$. Herein, a wedge direct sum $\hat{\oplus}$ is applied to $2\times 2\times 2$ patterns whenever they can be attached to each other.

Now, $\mathbb{X}_{2\times m_2\times 2}$ can be obtained as follows.

Theorem 2.1. *For any* $m_2 \geq 2$, $\Sigma_{2\times m_2\times 2} = \{y_{j_1 j_2\cdots j_{m_2}}\}$, *where* $y_{j_1 j_2\cdots j_{m_2}}$ *is given in Eq. (19). Furthermore, the ordering matrix* $\mathbb{X}_{2\times m_2\times 2} = [y_{j_1 j_2\ldots j_{m_2}}]$ *which is a* $2^{2m_2} \times 2^{2m_2}$ *matrix can be decomposed into following matrices*

$$\mathbb{X}_{2\times m_2\times 2} = [X_{2\times m_2\times 2;j_1}]_{2^2\times 2^2},$$

where $1 \leq j_1 \leq 2^4$. *For fixed* $j_1, j_2, \ldots, j_k \in \{1, 2, \ldots, 2^4\}$,

$$X_{2\times m_2\times 2;j_1 j_2\cdots j_k} = [X_{2\times m_2\times 2;j_1 j_2\cdots j_k j_{k+1}}]_{2^2\times 2^2},$$

where $1 \leq j_{k+1} \leq 2^4$ *and* $k \in \{1, 2, \ldots, m_2 - 2\}$. *For fixed* $j_1, j_2, \ldots, j_{m_2-1}$,

$$X_{2\times m_2\times 2;j_1 j_2\cdots j_{m_2-1}} = [y_{2\times m_2\times 2;j_1 j_2\cdots j_{m_2-1} j_{m_2}}]_{2^2\times 2^2},$$

where $y_{2\times m_2\times 2;j_1 j_2\cdots j_{m_2}}$ *is defined as in Eq. (19).*

Proof. From Eq. (12), $u_{\alpha_1\alpha_2\alpha_3}$ can be solved in terms of j_{α_2}, yielding

$$u_{1\alpha_2 1} = \left[\frac{j_{\alpha_2} - 1}{2^3}\right], \tag{20}$$

$$u_{1\alpha_2 2} = \left[\frac{j_{\alpha_2} - 1 - 2^3 u_{1\alpha_2 1}}{2^2}\right], \tag{21}$$

$$u_{2\alpha_2 1} = \left[\frac{j_{\alpha_2} - 1 - 2^3 u_{1\alpha_2 1} - 2^2 u_{1\alpha_2 2}}{2}\right], \tag{22}$$

$$u_{2\alpha_2 2} = j_{\alpha_2} - 1 - 2^3 u_{1\alpha_2 1} - 2^2 u_{1\alpha_2 2} - 2u_{2\alpha_2 1}, \tag{23}$$

where [ ] is the Gauss symbol. Equations (20)–(23), yield the following table.

j_{α_2}	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$u_{1\alpha_2 1}$	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1
$u_{1\alpha_2 2}$	0	0	0	0	1	1	1	1	0	0	0	0	1	1	1	1
$u_{2\alpha_2 1}$	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1
$u_{2\alpha_2 2}$	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1

For any $m_2 \geq 2$, we have

$$i_{m_2;1} = 1 + \sum_{\alpha_2=1}^{m_2} (2^{2(m_2-\alpha_2)+1} u_{1\alpha_2 1} + 2^{2(m_2-\alpha_2)} u_{1\alpha_2 2}),$$

$$i_{m_2;2} = 1 + \sum_{\alpha_2=1}^{m_2} (2^{2(m_2-\alpha_2)+1} u_{2\alpha_2 1} + 2^{2(m_2-\alpha_2)} u_{2\alpha_2 2}).$$

From the above formulae,

$$\begin{aligned} i_{m_2+1;1} &= 2^2(i_{m_2;1} - 1) + 2u_{1(m_2+1)1} + u_{1(m_2+1)2} + 1, \\ i_{m_2+1;2} &= 2^2(i_{m_2;2} - 1) + 2u_{2(m_2+1)1} + u_{2(m_2+1)2} + 1. \end{aligned}$$

Now, by induction on m_2 the theorem follows from the last two formulae and the above table. The proof is complete. ■

Remark 2.2. By the same method, the following relations can be derived. The details of the proof are omitted here for brevity.

$$\hat{\mathbb{X}}_{2\times 2\times m_3} = [z_{2\times 2\times m_3;k_1 k_2\ldots k_{m_3-1}k_{m_3}}]_{2^{2m_3}\times 2^{2m_3}}$$
$$\mathbb{Y}_{m_1\times 2\times 2} = [x_{m_1\times 2\times 2;i_1 i_2\ldots i_{m_1-1}i_{m_1}}]_{2^{2m_1}\times 2^{2m_1}}$$
$$\hat{\mathbb{Y}}_{2\times 2\times m_3} = [\hat{z}_{2\times 2\times m_3;\hat{k}_1\hat{k}_2\ldots\hat{k}_{m_3-1}\hat{k}_{m_3}}]_{2^{2m_3}\times 2^{2m_3}}$$
$$\mathbb{Z}_{m_1\times 2\times 2} = [\hat{x}_{m_1\times 2\times 2;\hat{i}_1\hat{i}_2\ldots\hat{i}_{m_1-1}\hat{i}_{m_1}}]_{2^{2m_1}\times 2^{2m_1}}$$
$$\hat{\mathbb{Z}}_{2\times m_2\times 2} = [\hat{y}_{2\times m_2\times 2;\hat{j}_1\hat{j}_2\ldots\hat{j}_{m_2-1}\hat{j}_{m_2}}]_{2^{2m_2}\times 2^{2m_2}}$$

Next, $[x]$-ordering is converted into $[\hat{x}]$-ordering for $\mathbf{Z}_{1\times m_2\times 2}$. Since $\mathbf{Z}_{1\times m_2\times 2} = \{(1, \alpha_2, \alpha_3) : 1 \leq \alpha_2 \leq m_2, 1 \leq \alpha_3 \leq 2\}$, the position (α_2, α_3) is the αth in Eq. (16), where

$$\alpha = 2(\alpha_2 - 1) + \alpha_3. \tag{24}$$

In Eq. (17), the position of $(1, \alpha_2, \alpha_3)$ is the $\hat{\alpha}$th, where

$$\hat{\alpha} = m_2(\alpha_3 - 1) + \alpha_2.$$

The relation

$$\hat{\alpha} = m_2\alpha + (1 - 2m_2)\left[\frac{\alpha - 1}{2}\right] + (1 - m_2),$$

or

$$\hat{\alpha} = k \quad \text{if } \alpha = 2k - 1,$$

and

$$\hat{\alpha} = m_2 + k \quad \text{if } \alpha = 2k,$$

$1 \leq k \leq m_2$ is easily verified.

Now, the ordering $[\hat{x}]$ in Eq. (17) on $\mathbf{Z}_{1\times m_2\times 2}$ can be extended to $\mathbf{Z}_{1\times m_2\times m_3}$ by Eq. (18). For a fixed m_2, $[\hat{x}]$-ordering on $\mathbf{Z}_{1\times m_2\times m_3}$ is clearly one-dimensional; it grows in the z-direction. Given ordering Eq. (18) on $\mathbf{Z}_{1\times m_2\times m_3}$, for $U = (u_{\alpha_1\alpha_2\alpha_3}) \in \Sigma_{2\times m_2\times m_3}$, denoted by

$$\hat{i}_{\alpha_1} = 1 + \sum_{\alpha_2=1}^{m_2}\sum_{\alpha_3=1}^{m_3} u_{\alpha_1\alpha_2\alpha_3} 2^{m_2(m_3-\alpha_3)+(m_2-\alpha_2)},$$

where $\alpha_1 = 1, 2$,

$$\psi_{\hat{x}}(U) = 2^{m_2 m_3}(\hat{i}_1 - 1) + \hat{i}_2.$$

Now, let $\hat{x}_{\hat{i}_1\hat{i}_2} = U = (u_{\alpha_1\alpha_2\alpha_3})$, yielding the new ordering matrix $\hat{\mathbb{X}}_{2\times m_2\times 2} = [\hat{x}_{2\times m_2\times 2;\hat{i}_1\hat{i}_2}]$ for $\Sigma_{2\times m_2\times 2}$. The relationship between $\mathbb{X}_{2\times m_2\times 2}$ and $\hat{\mathbb{X}}_{2\times m_2\times 2}$ is established before $\hat{\mathbb{X}}_{2\times m_2\times m_3}$ is constructed from $\hat{\mathbb{X}}_{2\times m_2\times 2}$ for $m_3 \geq 3$.

A conversion sequence of orderings can be obtained from Eqs. (16) and (17), where P_k represents the permutation of $\mathbb{N}_{2m} = \{1, 2, \ldots, 2m_2\}$ such that $P_k(k+1) = k$, $P_k(k) = k+1$ and the other numbers are fixed. P_k is also the permutation on $\mathbf{Z}_{1\times m_2\times 2}$ such that it exchanges k and $k+1$ while keeping the other positions fixed, i.e.

$\cdot$		$k+1$		$\cdot$	$\cdot$
$\cdot$		$\cdot$	k		$\cdot$

$\xrightarrow{P_k}$

$\cdot$	$\cdot$	k		$\cdot$	$\cdot$
$\cdot$	$\cdot$	$\cdot$		$k+1$	$\cdot$

.

Clearly, Eq. (16) can be converted into Eq. (17) in many ways using the sequence of P_k. A systematic approach is proposed here.

Lemma 2.3. *For $m_2 \geq 2$, Eq. (16) can be converted into Eq. (17) using the following sequences of $(m_2(m_2-1))/2$ permutations successively*

$$(P_2P_4\cdots P_{2m_2-2})(P_3P_5\cdots P_{2m_2-3})\cdots (P_kP_{k+2}\cdots P_{2m_2-k})\cdots(P_{m_2-1}P_{m_2+1})P_{m_2}, \tag{25}$$

$2 \leq k \leq m_2$.

Proof. When $m_2 = 2$ and 3, verifying that Eq. (25) can convert Eq. (16) into Eq. (17) is relatively simple.

When $m_2 \geq 4$, and for any $2 \leq k \leq m_2$, applying

$$(P_2P_4\cdots P_{2m_2-2})(P_3P_5\cdots P_{2m_2-3})\cdots (P_kP_{k+2}\cdots P_{2m_2-k})$$

to Eq. (16), yields two intermediate cases:

(i) When $2 \leq k \leq [m_2/2]$,

$k+1$	$k+3$	$\cdots$	$3k-1$	$\cdots$	$\cdots$	$\cdots$	$3k-1+2\ell$	$\cdots$	$2m_2-k-1$	$2m_2-k+1$	$\cdots$	$2m_2-1$	$2m_2$
1	2	$\cdots$	k	$k+2$	$k+4$	$\cdots$	$k+2\ell$	$\cdots$	$\cdots$	$2m_2-3k+1$	$\cdots$	$2m_2-k-2$	$2m_2-k$

, (26)

where $0 \leq \ell \leq m_2 - 2k$.

(ii) When $[m_2/2] + 1 \leq k \leq m_2 - 1$,

$k+1$	$\cdots$	$2m_2-k-1$	$2m_2-k+1$	$2m_2-k+2$	$\cdots$	$\cdots$	$\cdots$	$2m_2-1$	$2m_2$
1	2	$\cdots$	$\cdots$	$\cdots$	$k-1$	k	$k+2$	$\cdots$	$2m_2-k$

. (27)

When $k = m_2$ in Eq. (27), Eq. (17) holds. Equations (26) and (27) are established by mathematical induction on k. When $k = 2$, verifying that Eq. (16) is converted into

3	5	$\cdots$	$\cdots$	$\cdots$	$2m_2-3$	$2m_2-1$	$2m_2$
1	2	4	$\cdots$	$\cdots$	$\cdots$	$2m_2-4$	$2m_2-2$

by $P_2P_4\cdots P_{2m_2-2}$ is relatively easy such that Eq. (26) holds for $k = 2$. Next, assume that Eq. (26) holds for $k \leq [m/2]$. Then, by applying $P_{k+1}P_{k+3}\cdots P_{2m_2-k-1}$ to Eq. (26), Eq. (26) can be verified to hold for $k+1$ when $k+1 \leq [m_2/2]$ or becomes Eq. (27) when $k+1 \geq [m_2/2]$. When $k \geq [m_2/2]+1$, $P_{k+1}P_{k+3}\cdots P_{2m_2-k-1}$ is applied to Eq. (27). Equation (27) can also be confirmed to hold for $k+1$. Finally, Eq. (17) is concluded to hold for $k = m_2$. The proof is thus complete. ■

Based on Lemma 2.3, $\mathbb{X}_{2\times m_2\times 2}$ can be converted into $\hat{\mathbb{X}}_{2\times m_2\times 2}$ as follows. Let

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{28}$$

and for $2 \le j \le 2m_2 - 2$, as denoted by

$$P_{2m_2;j} = I_{2^{j-1}} \otimes P \otimes I_{2^{2m_2-j-1}},$$

where I_k is the $k \times k$ identity matrix. Moreover, let

$$\begin{aligned}\mathbb{P}_{x;2\times m_2\times 2} &= (P_{2m_2;2}P_{2m_2;4}\cdots P_{2m_2;2m_2-2}) \\ &\quad \cdots (P_{2m_2;k}\cdots P_{2m_2;2m_2-k})\cdots(P_{2m_2;m_2}),\end{aligned} \tag{29}$$

$2 \le k \le m_2$. Then, the following theorem holds.

Theorem 2.4. *For any $m_2 \ge 2$,*

$$\hat{\mathbb{X}}_{2\times m_2\times 2} = \mathbb{P}^t_{x;2\times m_2\times 2}\mathbb{X}_{2\times m_2\times 2}\mathbb{P}_{x;2\times m_2\times 2}. \tag{30}$$

Proof. From Eq. (24), in $\mathbf{Z}_{1\times m_2\times 2}$ the position (α_2, α_3) is the αth in Eq. (16), where $\alpha = 2(\alpha_2 - 1) + \alpha_3$. Define

$$\ell_\alpha \equiv 1 + 2u_{1\alpha_2\alpha_3} + u_{2\alpha_2\alpha_3},$$

$1 \le \ell_\alpha \le 4$ and $1 \le \alpha \le 2m_2$. For $U = (u_{\alpha_1\alpha_2\alpha_3}) \in \Sigma_{2\times m_2\times 2}$, from Theorem 2.1 it can be denoted by $y_{2\times m_2\times 2;j_1j_2\ldots j_{m_2}}$ and by Eq. (12) for fixed $1 \le \alpha_2 \le m_2$:

$$j_{\alpha_2} = 1 + 2^3u_{1\alpha_21} + 2^2u_{1\alpha_22} + 2u_{2\alpha_21} + u_{2\alpha_22},$$

where $1 \le j_{\alpha_2} \le 16$. Accordingly, $y_{j_{\alpha_2}}$ can be represented by $y_{\ell_{2\alpha_2-1}\ell_{2\alpha_2}}$ and the relation is

$$\begin{bmatrix} y_1 & y_2 & y_3 & y_4 \\ y_5 & y_6 & y_7 & y_8 \\ y_9 & y_{10} & y_{11} & y_{12} \\ y_{13} & y_{14} & y_{15} & y_{16} \end{bmatrix} = \begin{bmatrix} y_{11} & y_{12} & y_{21} & y_{22} \\ y_{13} & y_{14} & y_{23} & y_{24} \\ y_{31} & y_{32} & y_{41} & y_{42} \\ y_{33} & y_{34} & y_{43} & y_{44} \end{bmatrix}.$$

Therefore, from Eq. (19) patterns in ordering matrix $\mathbb{X}_{2\times m_2\times 2}$ can be specified by

$$\begin{aligned}y_{2\times m_2\times 2;j_1j_2\cdots j_{m_2}} &= y_{j_1} \oplus y_{j_2} \oplus \cdots \oplus y_{j_{m_2}} \\ &= y_{\ell_1\ell_2} \oplus y_{\ell_3\ell_4} \oplus \cdots \oplus y_{\ell_{2m_2-1}\ell_{2m_2}} \\ &\equiv y_{\ell_1\ell_2\cdots\ell_{2m_2}}.\end{aligned}$$

For any $1 \le k \le 2m_2 - 1$,

$$\begin{aligned}&P^t_{2m_2;k}\mathbb{X}_{2\times m_2\times 2}P_{2m_2;k} \\ &\quad = P^t_{2m_2;k}[y_{\ell_1\ell_2\cdots\ell_k\ell_{k+1}\cdots\ell_{2m_2}}]P_{2m_2;k} \\ &\quad = [y_{\ell_1\ell_2\cdots\ell_{k+1}\ell_k\cdots\ell_{2m_2}}]\end{aligned}$$

is easily verified, such that $P_{2m_2;k}$ exchanges ℓ_k and ℓ_{k+1} in $\mathbb{X}_{2\times m_2\times 2}$. Therefore, Eq. (30) follows from Eq. (29) and Lemma 2.3. ■

Now, according to Theorem 2.4,

$$\hat{\mathbb{X}}_{2\times m_2\times 2} = [\hat{x}_{2\times m_2\times 2;\hat{i}_1\hat{i}_2}],$$

$1 \le \hat{i}_1, \hat{i}_2 \le 2m_2$. From some observations as Eq. (15), $\hat{\mathbb{X}}_{2\times m_2\times 2}$ can be represented as $z_{2\times m_2\times 2;k_1k_2}$, where $1 \le k_1, k_2 \le 2^{2m_2}$. The $[\hat{x}]$-expression

$$\hat{\mathbb{X}}_{2\times m_2\times 2} = \mathbb{Z}^{(r)}_{2\times m_2\times 2} \tag{31}$$

for $\Sigma_{2\times m_2\times 2}$ enables $\hat{\mathbb{X}}_{2\times m_2\times m_3}$ to be constructed for $\Sigma_{2\times m_2\times m_3}$. Indeed, for fixed $m_2 \ge 2$ and $m_3 \ge 2$, let

$$\begin{aligned}\hat{x}_{2\times m_2\times m_3;\hat{i}_1\hat{i}_2} &\equiv z_{2\times m_2\times m_3;k_1k_2\cdots k_{m_3}} \\ &\equiv z_{2\times m_2\times 2;k_1k_2}\hat{\oplus}z_{2\times m_2\times 2;k_2k_3} \\ &\quad \hat{\oplus}\cdots\hat{\oplus}z_{2\times m_2\times 2;k_{m_3-1}k_{m_3}}.\end{aligned} \tag{32}$$

Therefore, by a similar argument as was used to establish Theorem 2.1 the following theorem holds for $\hat{\mathbb{X}}_{2\times m_2\times m_3}$, the detailed proofs are omitted for brevity.

Theorem 2.5. *For fixed $m_2 \ge 2$ and for any $m_3 \ge 2$, the ordering matrix $\hat{\mathbb{X}}_{2\times m_2\times m_3}$ with respect to $[\hat{x}]$-ordering can be expressed as*

$$\hat{\mathbb{X}}_{\hat{x};2\times m_2\times m_3} = [\hat{X}_{2\times m_2\times m_3;k_1}]_{2^{m_2}\times 2^{m_2}},$$

where $1 \le k_1 \le 2^{2m_2}$. For fixed $1 \le k_1, k_2, \ldots, k_l \le 2^{2m_2}$,

$$\hat{X}_{2\times m_2\times m_3;k_1k_2\cdots k_l} = [\hat{X}_{2\times m_2\times m_3;k_1k_2\cdots k_lk_{l+1}}]_{2^{m_2}\times 2^{m_2}}$$

where $1 \le k_{l+1} \le 2^{2m_2}$ and $1 \le l \le m_3 - 2$. For fixed $k_1, k_2, \ldots, k_{m_3-1}$,

$$\hat{X}_{2\times m_2\times m_3;k_1k_2\cdots k_{m_3-1}} = [z_{2\times m_2\times m_3;k_1k_2\cdots k_{m_3}}],$$

where $z_{2\times m_2\times m_3;k_1k_2\cdots k_{m_3}}$ is given by Eq. (32).

Remark 2.6. Similarly, according to other orderings, the following relations can be derived

$$\begin{aligned}\mathbb{X}_{2\times m_2\times m_3} &= [y_{2\times m_2\times m_3;j_1j_2\cdots j_{m_2}}]_{2^{m_2m_3}\times 2^{m_2m_3}} \\ \hat{\mathbb{Y}}_{m_1\times 2\times m_3} &= [\hat{z}_{m_1\times 2\times m_3;\hat{k}_1\hat{k}_2\cdots\hat{k}_{m_3}}]_{2^{m_1m_3}\times 2^{m_1m_3}} \\ \mathbb{Y}_{m_1\times 2\times m_3} &= [x_{m_1\times 2\times m_3;i_1i_2\cdots i_{m_1}}]_{2^{m_1m_3}\times 2^{m_1m_3}}\end{aligned}$$

$$\hat{\mathbb{Z}}_{m_1\times m_2\times 2} = [\hat{y}_{m_1\times m_2\times 2;\hat{j}_1\hat{j}_2\cdots\hat{j}_{m_2}}]_{2^{m_1m_2}\times 2^{m_1m_2}}$$
$$\mathbb{Z}_{m_1\times m_2\times 2} = [\hat{x}_{m_1\times m_2\times 2;\hat{i}_1\hat{i}_2\cdots\hat{i}_{m_1}}]_{2^{m_1m_2}\times 2^{m_1m_2}}.$$

3. Transition Matrices and Spatial Entropy

3.1. *Transition matrices*

Based on the definitions of the ordering matrices $\hat{\mathbb{X}}_{2\times m_2\times m_3}$ for $\Sigma_{2\times m_2\times m_3}$ having been defined, high order transition matrices $\mathbb{A}_{\hat{x};2\times m_2\times m_3}$ can now be derived from $\mathbb{A}_{x;2\times 2\times 2}$. As in the two-dimensional case [Ban & Lin, 2006], a basic set $\mathcal{B}\subset\Sigma_{2\times 2\times 2}$ is assumed be given. Define the transition matrix $\mathbb{A}_{x;2\times 2\times 2}=\mathbb{A}_{x;2\times 2\times 2}(\mathcal{B})$ by

$$\mathbb{A}_{x;2\times 2\times 2} = [a_{x;2\times 2\times 2;i_1i_2}]_{2^4\times 2^4}, \tag{33}$$

where

$$a_{x;2\times 2\times 2;i_1i_2} = \begin{cases} 1 & \text{if } x_{i_1i_2}\in\mathcal{B},\\ 0 & \text{otherwise.}\end{cases} \tag{34}$$

Then, the transition matrix $\mathbb{A}_{x;2\times m_2\times 2}$ is a $2^{2m_2}\times 2^{2m_2}$ matrix with entries $a_{x;2\times m_2\times 2;i_1i_2}$,[3] where

$$\begin{aligned} a_{x;2\times m_2\times 2;i_1i_2} &= a_{y;2\times m_2\times 2;j_1j_2\ldots j_{m_2}}\\ &= \prod_{k=1}^{m_2-1} a_{y;2\times 2\times 2;j_kj_{k+1}}. \end{aligned}\tag{35}$$

Before $\mathbb{A}_{x;2\times m_2\times 2}$ is introduced, three products of the matrices are defined as follows.

Definition 3.1. For any two matrices $\mathbb{M}=(M_{ij})$ and $\mathbb{N}=(N_{kl})$, the Kronecker product (tensor product) $\mathbb{M}\otimes\mathbb{N}$ of $\mathbb{M}$ and $\mathbb{N}$ is defined by

$$\mathbb{M}\otimes\mathbb{N} = (M_{ij}\mathbb{N}).$$

For any $n\geq 1$,

$$\otimes\mathbb{N}^n = \mathbb{N}\otimes\mathbb{N}\otimes\cdots\otimes\mathbb{N},$$

n-times in $\mathbb{N}$.

Next, for any two $m\times m$ matrices

$$\mathbb{P}=(P_{ij}) \quad\text{and}\quad \mathbb{Q}=(Q_{ij})$$

where P_{ij} and Q_{ij} are numbers or matrices, the Hadamard product $\mathbb{P}\circ\mathbb{Q}$ is defined by

$$\mathbb{P}\circ\mathbb{Q} = (P_{ij}\cdot Q_{ij}),$$

where the product $P_{ij}\cdot Q_{ij}$ of P_{ij} and Q_{ij} may be a multiplication of numbers, of numbers and matrices or of matrices whenever it is well-defined.

Finally, product $\hat{\otimes}$ is defined as follows. For any 4×4 matrix

$$\mathbb{M}_2 = \begin{bmatrix} m_{11} & m_{12} & m_{21} & m_{22}\\ m_{13} & m_{14} & m_{23} & m_{24}\\ m_{31} & m_{32} & m_{41} & m_{42}\\ m_{33} & m_{34} & m_{43} & m_{44}\end{bmatrix} = \begin{bmatrix} M_{2;1} & M_{2;2}\\ M_{2;3} & M_{2;4}\end{bmatrix}$$

and any 2×2 matrix

$$\mathbb{N} = \begin{bmatrix} N_1 & N_2\\ N_3 & N_4\end{bmatrix},$$

where m_{ij} are numbers and N_k are numbers or matrices, for $1\leq i,j,k\leq 4$, define

$$\mathbb{M}_2\hat{\otimes}\mathbb{N} = \begin{bmatrix} m_{11}N_1 & m_{12}N_2 & m_{21}N_1 & m_{22}N_2\\ m_{13}N_3 & m_{14}N_4 & m_{23}N_3 & m_{24}N_4\\ m_{31}N_1 & m_{32}N_2 & m_{41}N_1 & m_{42}N_2\\ m_{33}N_3 & m_{34}N_4 & m_{43}N_3 & m_{44}N_4\end{bmatrix}.$$

Furthermore, for $n\geq 1$, the $n+1$-th order of transition matrix of $\mathbb{M}_2$ is defined by

$$\mathbb{M}_{n+1}\equiv\hat{\otimes}\mathbb{M}_2^n = \mathbb{M}_2\hat{\otimes}\mathbb{M}_2\hat{\otimes}\cdots\hat{\otimes}\mathbb{M}_2,$$

n-times in $\mathbb{M}_2$. More precisely,

$$\begin{aligned}\mathbb{M}_{n+1} &= \mathbb{M}_2\hat{\otimes}(\hat{\otimes}\mathbb{M}_2^{n-1})\\ &= \begin{bmatrix} M_{2;1}\circ(\hat{\otimes}\mathbb{M}_2^{n-1}) & M_{2;2}\circ(\hat{\otimes}\mathbb{M}_2^{n-1})\\ M_{2;3}\circ(\hat{\otimes}\mathbb{M}_2^{n-1}) & M_{2;4}\circ(\hat{\otimes}\mathbb{M}_2^{n-1})\end{bmatrix}\\ &= \left[\begin{array}{cc|cc} m_{11}M_{n;1} & m_{12}M_{n;2} & m_{21}M_{n;1} & m_{22}M_{n;2}\\ m_{13}M_{n;3} & m_{14}M_{n;4} & m_{23}M_{n;3} & m_{24}M_{n;4}\\ \hline m_{31}M_{n;1} & m_{32}M_{n;2} & m_{41}M_{n;1} & m_{42}M_{n;2}\\ m_{33}M_{n;3} & m_{34}M_{n;4} & m_{43}M_{n;3} & m_{44}M_{n;4}\end{array}\right]\\ &= \begin{bmatrix} M_{n+1;1} & M_{n+1;2}\\ M_{n+1;3} & M_{n+1;4}\end{bmatrix},\end{aligned}$$

where

$$\mathbb{M}_n = \hat{\otimes}\mathbb{M}_2^{n-1} = \begin{bmatrix} M_{n;1} & M_{n;2}\\ M_{n;3} & M_{n;4}\end{bmatrix}.$$

Here, the following convention is adopted,

$$\hat{\otimes}\mathbb{M}_2^0 = \mathbb{E}_2,$$

where $\mathbb{E}_2$ is the 2×2 matrix with 1 as its entries.

Theorem 2.1, yields results for $\mathbb{A}_{x;2\times m_2\times 2}$ as $\mathbb{T}_n$ in Theorem 3.1 in [Ban & Lin, 2006]. Indeed,

[3]Use $a_{x;2\times 2\times 2;i_1i_2}$ to substitute $a_{x;2\times 2\times 2;i_1,i_2}$ for simplicity afterward.

Theorem 3.2. *Let $\mathbb{A}_{x;2\times 2\times 2}$ be a transition matrix that is given by Eqs. (33) and (34). Then, for high order transition matrices $\mathbb{A}_{x;2\times m_2\times 2}$, $m_2 \geq 3$, the following three equivalent statements hold:*

(I) *$\mathbb{A}_{x;2\times m_2\times 2}$ can be decomposed into m_2 successive 4×4 matrices*

$$\mathbb{A}_{x;2\times m_2\times 2} = [A_{x;2\times m_2\times 2;j_1}]_{4\times 4},$$

where $1 \leq j_1 \leq 16$. For fixed $1 \leq j_1, j_2, \ldots, j_k \leq 16$,

$$A_{x;2\times m_2\times 2;j_1 j_2\cdots j_k} = [A_{x;2\times m_2\times 2;j_1 j_2\cdots j_k j_{k+1}}]_{4\times 4},$$

where $1 \leq j_{k+1} \leq 16$ and $1 \leq k \leq m_2 - 1$. For fixed $j_1, j_2, \ldots, j_{m_2-1} \in \{1, 2, \ldots, 16\}$,

$$A_{x;2\times m_2\times 2;j_1 j_2\cdots j_{m_2-1}} = [a_{y;2\times m_2\times 2;j_1 j_2\cdots j_{m_2}}]_{4\times 4},$$

where $a_{y;2\times m_2\times 2;j_1 j_2\ldots j_{m_2}}$ is defined in Eq. (35).

(II) *Starting from*

$$\mathbb{A}_{x;2\times 2\times 2} = [A_{x;2\times 2\times 2;j_1}]_{4\times 4}$$

and

$$A_{x;2\times 2\times 2;j_1} = [a_{y;2\times 2\times 2;j_1 j_2}]_{4\times 4},$$

for $m_2 \geq 3$, $\mathbb{A}_{x;2\times m_2\times 2}$ can be obtained from $\mathbb{A}_{x;2\times (m_2-1)\times 2}$ by replacing $A_{x;2\times 2\times 2;j_1}$ with

$$(A_{x;2\times 2\times 2;j_1})_{4\times 4} \circ (\mathbb{A}_{x;2\times 2\times 2})_{4\times 4}.$$

(III) *For $m_2 \geq 3$,*

$$\mathbb{A}_{x;2\times m_2\times 2} = (\mathbb{A}_{x;2\times (m_2-1)\times 2})_{2^{2(m_2-1)}\times 2^{2(m_2-1)}} \circ (E_{2^{2(m_2-2)}} \otimes \mathbb{A}_{x;2\times 2\times 2}), \quad (36)$$

where E_{2^k} is the $2^k \times 2^k$ matrix with 1 as its entries.

Proof

(I) The proof involves simply replacing $X_{2\times m_2\times 2;j_1 j_2\cdots j_k}$ and $y_{2\times m_2\times 2;j_1 j_2\cdots j_{m_2}}$ by $A_{x;2\times m_2\times 2;j_1 j_2\cdots j_k}$ and $a_{y;2\times m_2\times 2;j_1 j_2\cdots j_{m_2}}$ in Theorem 2.1, respectively.

(II) follows directly from (I).

(III) follows from (I); $\mathbb{A}_{x;2\times m_2\times 2} = [A_{x;2\times m_2\times 2;j_1}]$, $1 \leq j_1 \leq 2^4$. (I) yields the following formula;

$$\begin{aligned}\mathbb{A}_{x;2\times m_2\times 2} &= [a_{y;2\times 2\times 2;j_1 j_2} A_{x;2\times (m_2-1)\times 2;j_2}] \\ &= (\mathbb{A}_{x;2\times (m_2-1)\times 2})_{2^{2(m_2-1)}\times 2^{2(m_2-1)}} \\ &\quad \hat{\otimes}\, [E_{2^{2(m_2-2)}} \otimes \mathbb{A}_{x;2\times 2\times 2}].\end{aligned}$$

The proof is complete. ■

Remark 3.3. As stated in Remark 2.2, the following formulae apply

$$\begin{aligned}\mathbb{A}_{\hat{x};2\times 2\times m_3} &= [a_{z;2\times 2\times m_3;k_1 k_2\cdots k_{m_3-1}k_{m_3}}]_{2^{2m_3}\times 2^{2m_3}} \\ \mathbb{A}_{y;m_1\times 2\times 2} &= [a_{x;m_1\times 2\times 2;i_1 i_2\cdots i_{m_1-1} i_{m_1}}]_{2^{2m_1}\times 2^{2m_1}} \\ \mathbb{A}_{\hat{y};2\times 2\times m_3} &= [a_{\hat{z};2\times 2\times m_3;\hat{k}_1\hat{k}_2\cdots \hat{k}_{m_3-1}\hat{k}_{m_3}}]_{2^{2m_3}\times 2^{2m_3}} \\ \mathbb{A}_{z;m_1\times 2\times 2} &= [a_{\hat{x};m_1\times 2\times 2;\hat{i}_1\hat{i}_2\cdots \hat{i}_{m_1-1}\hat{i}_{m_1}}]_{2^{2m_1}\times 2^{2m_1}} \\ \mathbb{A}_{\hat{z};2\times m_2\times 2} &= [a_{\hat{y};2\times m_2\times 2;\hat{j}_1\hat{j}_2\cdots \hat{j}_{m_2-1}\hat{j}_{m_2}}]_{2^{2m_2}\times 2^{2m_2}}.\end{aligned}$$

Now, the transition matrix $\mathbb{A}_{\hat{x};2\times m_2\times 2}$, with respect to the ordering matrix $\hat{\mathbb{X}}_{2\times m_2\times 2}$ can be obtained. Additionally, by using Theorem 2.4 yields

Theorem 3.4

$$\mathbb{A}_{\hat{x};2\times m_2\times 2} = \mathbb{P}^t_{x;2\times m_2\times 2}\mathbb{A}_{x;2\times m_2\times 2}\mathbb{P}_{x;2\times m_2\times 2}.$$

Proof. The proof involves simply replacing $y_{2\times m_2\times 2;j_1 j_2\cdots j_{m_2}}$ by $a_{y;2\times m_2\times 2;j_1 j_2\cdots j_{m_2}}$ in Theorem 2.4. ■

Theorem 2.5 yields transition matrix $\mathbb{A}_{\hat{x};2\times m_2\times m_3}$ from $\mathbb{A}_{\hat{x};2\times m_2\times 2}$. Equation (31) yields the transition matrix

$$\mathbb{A}_{\hat{x};2\times m_2\times 2} = [A_{\hat{x};2\times m_2\times 2;k_1}] \quad (37)$$

and

$$A_{\hat{x};2\times m_2\times 2;k_1} = [a_{z;2\times m_2\times 2;k_1 k_2}]. \quad (38)$$

Therefore,

Theorem 3.5. *Let $\mathbb{A}_{\hat{x};2\times m_2\times 2}$ be a transition matrix given by Eqs. (37) and (38). Then, for high order transition matrices $\mathbb{A}_{\hat{x};2\times m_2\times m_3}$, $m_2 \geq 3$, we have the following three equivalent statements hold,*

(I) *$\mathbb{A}_{\hat{x};2\times m_2\times m_3}$ can be decomposed into m_3 successive $2^{m_2} \times 2^{m_2}$ matrices:*

$$\mathbb{A}_{\hat{x};2\times m_2\times m_3} = [A_{\hat{x};2\times m_2\times m_3;k_1}]_{2^{m_2}\times 2^{m_2}},$$

where $1 \leq k_1 \leq 2^{2m_2}$. For fixed $1 \leq k_1, k_2, \ldots, k_\ell \leq 2^{2m_2}$,

$$A_{\hat{x};2\times m_2\times m_3;k_1 k_2\cdots k_\ell} = [A_{\hat{x};2\times m_2\times m_3;k_1 k_2\cdots k_\ell k_{\ell+1}}]_{2^{m_2}\times 2^{m_2}},$$

where $1 \leq k_{\ell+1} \leq 2^{2m_2}$ and $1 \leq \ell \leq m_3 - 2$,

$$A_{\hat{x};2\times m_2\times m_3;k_1 k_2\cdots k_{m_3-1}} = [a_{z;2\times m_2\times m_3;k_1 k_2\cdots k_{m_3}}]_{2^{m_2}\times 2^{m_2}},$$

where $1 \leq k_{m_3} \leq 2^{2m_2}$ and by Eq. (32)

$$a_{z;2\times m_2\times m_3;k_1 k_2\cdots k_{m_3}} = \prod_{\ell=1}^{m_3-1} a_{z;2\times m_2\times 2;k_\ell k_{\ell+1}}.$$

(II) *For any* $m_3 \geq 3$, $\mathbb{A}_{\hat{x};2\times m_2\times m_3}$ *can be obtained from* $\mathbb{A}_{\hat{x};2\times m_2\times(m_3-1)}$ *by replacing* $A_{\hat{x};2\times m_2\times 2;k_1}$ *with*

$$(A_{\hat{x};2\times m_2\times 2;k_1})_{2^{m_2}\times 2^{m_2}} \circ (\mathbb{A}_{\hat{x};2\times m_2\times 2})_{2^{m_2}\times 2^{m_2}}.$$

(III) *Furthermore, for* $m_3 \geq 3$,

$$\begin{aligned}\mathbb{A}_{\hat{x};2\times m_2\times m_3} &= (\mathbb{A}_{\hat{x};2\times m_2\times(m_3-1)})_{2^{m_2(m_3-1)}\times 2^{m_2(m_3-1)}} \\ &\quad \circ (E_{2^{m_2(m_3-2)}} \otimes \mathbb{A}_{\hat{x};2\times m_2\times 2}). \end{aligned} \tag{39}$$

The proof closely resembles that of Theorems 2.1 and 3.2. Details of the proof are obvious and repeated, hence can be omitted.

Remark 3.6. As in Remark 2.6, the following formulae are obtained

$$\begin{aligned}
\mathbb{A}_{x;2\times m_2\times m_3} &= [a_{y;2\times m_2\times m_3;j_1j_2\cdots j_{m_2}}]_{2^{m_2m_3}\times 2^{m_2m_3}} \\
\mathbb{A}_{\hat{y};m_1\times 2\times m_3} &= [a_{\hat{z};m_1\times 2\times m_3;\hat{k}_1\hat{k}_2\cdots\hat{k}_{m_3}}]_{2^{m_1m_3}\times 2^{m_1m_3}} \\
\mathbb{A}_{y;m_1\times 2\times m_3} &= [a_{x;m_1\times 2\times m_3;i_1i_2\cdots i_{m_1}}]_{2^{m_1m_3}\times 2^{m_1m_3}} \\
\mathbb{A}_{\hat{z};m_1\times m_2\times 2} &= [a_{\hat{y};m_1\times m_2\times 2;\hat{j}_1\hat{j}_2\cdots\hat{j}_{m_2}}]_{2^{m_1m_2}\times 2^{m_1m_2}} \\
\mathbb{A}_{z;m_1\times m_2\times 2} &= [a_{\hat{x};m_1\times m_2\times 2;\hat{i}_1\hat{i}_2\cdots\hat{i}_{m_1}}]_{2^{m_1m_2}\times 2^{m_1m_2}}.
\end{aligned}$$

Finally, the spatial entropy $h(\mathcal{B})$ can be computed from the maximum eigenvalue $\lambda_{\hat{x};2,m_2,m_3}$ of $\mathbb{A}_{\hat{x};2\times m_2\times m_3}$. Indeed,

Theorem 3.7. *Let* $\lambda_{\hat{x};2,m_2,m_3}$ *be the maximum eigenvalue of* $\mathbb{A}_{\hat{x};2\times m_2\times m_3}$, *then*

$$h(\mathcal{B}) = \lim_{m_2,m_3\to\infty} \frac{\log \lambda_{\hat{x};2,m_2,m_3}}{m_2m_3}. \tag{40}$$

Proof. By the same arguments as in [Chow *et al.*, 1996a], the limit Eq. (1) is well-defined and exists. From $\mathbb{A}_{\hat{x};2\times m_2\times m_3}$, for $m_2 \geq 2$ and $m_3 \geq 2$,

$$\begin{aligned}\Gamma_{\hat{x};m_1\times m_2\times m_3}(\mathcal{B}) &= \sum_{1\leq i,j\leq 2^{m_2m_3}} (\mathbb{A}^{m_1-1}_{\hat{x};2\times m_2\times m_3})_{ij} \\ &= |(\mathbb{A}^{m_1-1}_{\hat{x};2\times m_2\times m_3})|.\end{aligned}$$

As in the one-dimensional case,

$$\lim_{m_1\to\infty} \frac{\log|(\mathbb{A}^{m_1-1}_{\hat{x};2\times m_2\times m_3})|}{m_1} = \log \lambda_{\hat{x};2,m_2,m_3},$$

as for example [Ban & Lin, 2005]. Hence,

$$\begin{aligned}h(\mathcal{B}) &= \lim_{m_1,m_2,m_3\to\infty} \frac{\log \Gamma_{\hat{x};m_1\times m_2\times m_3}(\mathcal{B})}{m_1m_2m_3} \\ &= \lim_{m_2,m_3\to\infty} \frac{1}{m_2m_3}\left(\lim_{m_1\to\infty} \frac{\log \Gamma_{\hat{x};m_1\times m_2\times m_3}(\mathcal{B})}{m_1}\right) \\ &= \lim_{m_2,m_3\to\infty} \frac{\log \lambda_{\hat{x};2,m_2,m_3}}{m_2m_3}.\end{aligned}$$

The proof is complete. ■

Remark 3.8. Let $\lambda_{x;2,m_2,m_3}$, $\lambda_{\hat{y};m_1,2,m_3}$, $\lambda_{y;m_1,2,m_3}$, $\lambda_{\hat{z};m_1,m_2,2}$ and $\lambda_{z;m_1,m_2,2}$ be the maximum eigenvalue of $\mathbb{A}_{x;2\times m_2\times m_3}$, $\mathbb{A}_{\hat{y};m_1\times 2\times m_3}$, $\mathbb{A}_{y;m_1\times 2\times m_3}$, $\mathbb{A}_{\hat{z};m_1\times m_2\times 2}$ and $\mathbb{A}_{z;m_1\times m_2\times 2}$ respectively. Then,

$$\begin{aligned}h(\mathcal{B}) &= \lim_{m_2,m_3\to\infty} \frac{\log \lambda_{x;2,m_2,m_3}}{m_2m_3} \\ &= \lim_{m_1,m_3\to\infty} \frac{\log \lambda_{\hat{y};m_1,2,m_3}}{m_1m_3} \\ &= \lim_{m_1,m_3\to\infty} \frac{\log \lambda_{y;m_1,2,m_3}}{m_1m_3} \\ &= \lim_{m_1,m_2\to\infty} \frac{\log \lambda_{\hat{z};m_1,m_2,2}}{m_1m_2} \\ &= \lim_{m_1,m_2\to\infty} \frac{\log \lambda_{z;m_1,m_2,2}}{m_1m_2}.\end{aligned}$$

The detailed proofs are as above.

3.2. *Computation of $\lambda_{m,n}$ and entropies*

The last subsection provided a systematic means of writing down $\mathbb{A}_{\hat{x};2\times m_2\times m_3}$ from $\mathbb{A}_{x;2\times 2\times 2}$. As in a two-dimensional case [Ban & Lin, 2005], a recursive formula for $\lambda_{\hat{x};2,m_2,m_3}$ can be obtained in a special structure. An illustrative example is presented in which $\mathbb{A}_{\hat{x};2\times m_2\times m_3}$ and $\lambda_{\hat{x};2,m_2,m_3}$ can be derived explicitly to demonstrate the methods developed in the preceding subsection. More complete results will be presented later.

Let

$$G = \begin{bmatrix}1 & 1\\ 1 & 0\end{bmatrix} \quad \text{and} \quad E = E_2 = \begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}, \tag{41}$$

and

$$\mathbb{A}_{x;2\times 2\times 2} = G \otimes E \otimes E \otimes E. \tag{42}$$

Proposition 3.9. *Substitute* $\mathbb{A}_{x;2\times 2\times 2}$ *into Eqs. (41) and (42). Then,*

(i) $\mathbb{A}_{x;2\times m_2\times 2} = \otimes(G\otimes E)^{m_2-1} \otimes (\otimes E^2)$, (43)

(ii) $\mathbb{A}_{\hat{x};2\times m_2\times 2} = (\otimes G^{m_2-1}) \otimes (\otimes E^{m_2+1})$, (44)

(iii) $\mathbb{A}_{\hat{x};2\times m_2\times m_3} = \otimes((\otimes G^{m_2-1})\otimes E)^{m_3-1} \otimes (\otimes E^{m_2})$. (45)

Furthermore, for the maximum eigenvalue $\lambda_{\hat{x};2,m_2,m_3}$ *of* $\mathbb{A}_{\hat{x};2\times m_2\times m_3}$, *the following recursive formulae apply*:

$$\lambda_{\hat{x};2,m_2+1,m_3} = 2g^{m_3-1}\lambda_{\hat{x};2,m_2,m_3} \tag{46}$$

and

$$\lambda_{\hat{x};2,m_2,m_3+1} = 2g^{m_2-1}\lambda_{\hat{x};2,m_2,m_3} \tag{47}$$

for $m_2, m_3 \geq 2$ *with*

$$\lambda_{\hat{x};2,2,2} = 2^3 g. \tag{48}$$

The spatial entropy is

$$h(\mathbb{A}_{x;2\times 2\times 2}) = \log g, \tag{49}$$

where $g = (1+\sqrt{5})/2$, *the golden-mean.*

Proof. The proof is described only briefly, and the details are omitted for brevity.

(i) can be proven by Theorem 3.2 and induction on m_2. Indeed, by Eq. (36),

$$\begin{aligned}
\mathbb{A}_{x;2\times 3\times 2} &= (\mathbb{A}_{x;2\times 2\times 2})_{4\times 4} \circ (E_{2^2} \otimes \mathbb{A}_{x;2\times 2\times 2})_{4\times 4}\\
&= (G\otimes E\otimes E\otimes E)_{4\times 4} \circ (E\otimes E\otimes (G\otimes E\otimes E\otimes E))_{4\times 4}\\
&= (G\circ E)\otimes (E\circ E)\otimes (E\circ G)\otimes (E_{2\times 2}\circ (E\otimes E\otimes E))_{2\times 2}\\
&= G\otimes E\otimes G\otimes E\otimes E\otimes E.
\end{aligned}$$

Assume that $\mathbb{A}_{x;2\times (m_2-1)\times 2} = \otimes(G\otimes E)^{m_2-2}\otimes(\otimes E^2)$. Then by Eq. (36) again,

$$\begin{aligned}
\mathbb{A}_{x;2\times m_2\times 2} &= (\mathbb{A}_{x;2\times(m_2-1)\times 2}) \circ ((\otimes E^{2(m_2-2)})\otimes \mathbb{A}_{x;2\times 2\times 2})\\
&= (\otimes(G\otimes E)^{m_2-2}\otimes(\otimes E^2))_{2^{2m_2-2}\times 2^{2m_2-2}} \circ ((\otimes E^{2(m_2-2)})\otimes (G\otimes E\otimes E\otimes E)_{2^{2m_2-2}\times 2^{2m_2-2}}\\
&= (\otimes(G\otimes E)^{m_2-2}\otimes (E\otimes E))_{2^{2m_2-2}\times 2^{2m_2-2}}\\
&\quad \circ (\otimes(E\otimes E)^{m_2-2}\otimes (G\otimes E)\otimes (E\otimes E))_{2^{2m_2-2}\times 2^{2m_2-2}}\\
&= \otimes[(G\circ E)\otimes(E\circ E)]^{m_2-2}\otimes(E\circ G)\otimes(E\circ(E\otimes E\otimes E))\\
&= \otimes(G\otimes E)^{m_2-2}\otimes(G\otimes E)\otimes(E\otimes E)\\
&= \otimes(G\otimes E)^{m_2-1}\otimes(\otimes E^2).
\end{aligned}$$

(ii) The following property of matrices is required and detailed proofs are omitted: For any two 2×2 matrices A and B,

$$P(A\otimes B)P = B\otimes A, \tag{50}$$

where P is given by Eq. (28). Equation (44) is proven by induction on m_2. When $m_2 = 2$, by Theorem 3.1,

$$\begin{aligned}
\mathbb{A}_{\hat{x};2\times 2\times 2} &= \mathbb{P}^t_{x;2\times 2\times 2}\mathbb{A}_{x;2\times 2\times 2}\mathbb{P}_{x;2\times 2\times 2}\\
&= (P_{4;2})^t\mathbb{A}_{x;2\times 2\times 2}P_{4;2}\\
&= (I_2\otimes P\otimes I_2)((G\otimes E)\otimes(E\otimes E))(I_2\otimes P\otimes I_2)\\
&= G\otimes(P(E\otimes E)P)\otimes E\\
&= G\otimes E\otimes E\otimes E
\end{aligned}$$

by Eq. (50).

Now, Eq. (44) is assumed to hold for m_2-1;

$$\mathbb{A}_{\hat{x};2\times(m_2-1)\times 2} = (\otimes G^{m_2-2})\otimes(\otimes E^{m_2}).$$

Then

$$\begin{aligned}
\mathbb{A}_{\hat{x};2\times m_2\times 2} &= \mathbb{P}^t_{x;2\times m_2\times 2}\mathbb{A}_{x;2\times m_2\times 2}\mathbb{P}_{x;2\times m_2\times 2}\\
&= [(P_{2m_2;2}P_{2m_2;4}\cdots P_{2m_2;2m_2-2})(P_{2m_2;3}P_{2m_2;5}\cdots P_{2m_2;2m_2-3})\cdots(P_{2m_2;m_2})]^t\\
&\quad \mathbb{A}_{x;2\times m_2\times 2}[(P_{2m_2;2}P_{2m_2;4}\cdots P_{2m_2;2m_2-2})(P_{2m_2;3}P_{2m_2;5}\cdots P_{2m_2;2m_2-3})\cdots(P_{2m_2;m_2})]
\end{aligned}$$

$$= (P_{2m_2;m_2}) \cdots (P_{2m_2;3}P_{2m_2;5} \cdots P_{2m_2;2m_2-3})[(P_{2m_2;2}P_{2m_2;4} \cdots P_{2m_2;2m_2-2})$$
$$(\otimes(G \otimes E)^{m_2-1} \otimes (\otimes E^2))(P_{2m_2;2}P_{2m_2;4} \cdots P_{2m_2;2m_2-2})]$$
$$\times(P_{2m_2;3}P_{2m_2;5} \cdots P_{2m_2;2m_2-3}) \cdots (P_{2m;m_2})$$
$$= (P_{2m_2;m_2}) \cdots (P_{2m_2;3}P_{2m_2;5} \cdots P_{2m_2;2m_2-3})[G \otimes (\otimes(G \otimes E)^{m_2-2} \otimes (\otimes E^2)) \otimes E]$$
$$(P_{2m_2;3}P_{2m_2;5} \cdots P_{2m_2;2m_2-3}) \cdots (P_{2m_2;m_2})$$
$$= G \otimes \{(P_{2(m_2-1);m_2-1}) \cdots (P_{2(m_2-1);2}P_{2(m_2-1);4} \cdots P_{2(m_2-1);2(m_2-1)-2})[\otimes(G \otimes E)^{m_2-1}]$$
$$(P_{2(m_2-1);2}P_{2(m_2-1);4} \cdots P_{2(m_2-1);2(m_2-1)-2}) \cdots (P_{2(m_2-1);m_2-1})\} \otimes E$$
$$= G \otimes (\mathbb{P}^t_{x;2\times(m_2-1)\times 2}\mathbb{A}_{x;2\times(m_2-1)\times 2}\mathbb{P}_{x;2\times(m_2-1)\times 2}) \otimes E$$
$$= G \otimes \mathbb{A}_{\hat{x};2\times(m_2-1)\times 2} \otimes E$$
$$= G \otimes ((\otimes G^{m_2-2}) \otimes (\otimes E^{m_2})) \otimes E$$
$$= (\otimes G^{m_2-1}) \otimes (\otimes E^{m_2+1}).$$

(iii) For a fixed m_2, these results are proven by induction on $m_3 \geq 2$. Assume that Eq. (45) holds for $m_3 - 1$;

$$\mathbb{A}_{\hat{x};2\times m_2\times(m_3-1)} = \otimes((\otimes G^{m_2-1}) \otimes E)^{m_3-2} \otimes (\otimes E^{m_2}).$$

Then, by Eq. (39),

$$\begin{aligned}\mathbb{A}_{\hat{x};2\times m_2\times m_3} &= \mathbb{A}_{\hat{x};2\times m_2\times(m_3-1)} \circ ((\otimes E^{m_2(m_3-2)}) \otimes \mathbb{A}_{\hat{x};2\times m_2\times 2})\\ &= [\otimes((\otimes G^{m_2-1}) \otimes E)^{m_3-2} \otimes (\otimes E^{m_2})] \circ [(\otimes E^{m_2(m_3-2)}) \otimes (\otimes G^{m_2-1}) \otimes (\otimes E^{m_2+1})]\\ &= \otimes((\otimes G^{m_2-1}) \otimes E)^{m_3-2} \otimes ((\otimes G^{m_2-1}) \otimes (\otimes E^{m_2+1}))\\ &= \otimes((\otimes G^{m_2-1}) \otimes E)^{m_3-1} \otimes (\otimes E^{m_2}).\end{aligned}$$

For the maximum eigenvalue $\lambda_{\hat{x};2,m_2,m_3}$, Eq. (48) is easily verified. Equation (46) is established for fixed m_3 using Eq. (45), yielding

$$\begin{aligned}&\mathbb{A}_{\hat{x};2\times(m_2+1)\times m_3}\\ &\quad = \otimes((\otimes G^{m_2}) \otimes E)^{m_3-1} \otimes (\otimes E^{m_2+1})\\ &\quad = (G \otimes (\otimes G^{m_2-1}) \otimes E)^{m_3-1} \otimes (\otimes E^{m_2} \otimes E),\end{aligned}$$

which implies

$$\lambda_{\hat{x};2,m_2+1,m_3} = 2g^{m_3-1}\lambda_{\hat{x};2,m_2,m_3},$$

see [Bellman, 1970; Gantmacher, 1959] and [Horn & Johnson, 1990].

Similarly, for a fixed m_2, Eq. (47) is proven using Eq. (45) again:

$$\begin{aligned}&\mathbb{A}_{\hat{x};2\times m_2\times(m_3+1)}\\ &\quad = \otimes((\otimes G^{m_2-1}) \otimes E)^{m_3} \otimes (\otimes E^{m_2})\\ &\quad = \otimes((\otimes G^{m_2-1}) \otimes E) \otimes \mathbb{A}_{\hat{x};2\times m_2\times m_3},\end{aligned}$$

which implies

$$\lambda_{\hat{x};2,m_2,m_3+1} = 2g^{m_2-1}\lambda_{\hat{x};2,m_2,m_3}.$$

Finally, Eq. (49) follows from Eqs. (46) and (47) and Theorem 3.7. The proof is thus complete. ■

4. Connecting Operator

This section introduces the connecting operator and employs it to derive a recursive formula between an elementary pattern of order $(m_1, m_2, m_3 + 1)$ and that of order (m_1, m_2, m_3). It is also adopted to obtain a lower bound on entropy.

4.1. *Connecting operator in z-direction*

This subsection derives connecting operators and studies their properties. For brevity, only the connecting operator in the z-direction is discussed but the other cases are similar, and will be considered in the following remarks. For clarity, as in the former section, two symbols on lattice $\mathbf{Z}_{2\times 2\times 2}$ are examined first.

According to Theorem 3.5, the transition matrix $\mathbb{A}_{\hat{x};2\times m_2\times m_3}$ can be represented as $A_{\hat{x};2\times m_2\times m_3;\alpha}$, where $1 \leq \alpha \leq 2^{2m_2}$, is a $2^{m_2(m_3-1)} \times 2^{m_2(m_3-1)}$ matrix.

For matrix multiplication, the indices of $\mathbb{A}_{\hat{x};2\times m_2\times m_3}$ are conveniently expressed as

$$\begin{bmatrix} A_{\hat{x};2\times m_2\times m_3;11} & A_{\hat{x};2\times m_2\times m_3;12} & \cdots & A_{\hat{x};2\times m_2\times m_3;12^{m_2}} \\ A_{\hat{x};2\times m_2\times m_3;21} & A_{\hat{x};2\times m_2\times m_3;22} & \cdots & A_{\hat{x};2\times m_2\times m_3;22^{m_2}} \\ \vdots & \vdots & \ddots & \vdots \\ A_{\hat{x};2\times m_2\times m_3;2^{m_2}1} & A_{\hat{x};2\times m_2\times m_3;2^{m_2}2} & \cdots & A_{\hat{x};2\times m_2\times m_3;2^{m_2}2^{m_2}} \end{bmatrix}.$$

Clearly, $A_{\hat{x};2\times m_2\times m_3;\alpha} = A_{\hat{x};2\times m_2\times m_3;\beta_1\beta_2}$, where $\alpha = \alpha(\beta_1, \beta_2) = 2^{m_2}(\beta_1 - 1) + \beta_2$. For $m_1 \geq 2$, the elementary pattern in the entries of $\mathbb{A}^{m_1}_{\hat{x};2\times m_2\times m_3}$ is given by

$$A_{\hat{x};2\times m_2\times m_3;\beta_1\beta_2}A_{\hat{x};2\times m_2\times m_3;\beta_2\beta_3} \cdots A_{\hat{x};2\times m_2\times m_3;\beta_{m_1}\beta_{m_1+1}}$$

where $\beta_r \in \{1, 2, \ldots, 2^{m_2}\}$ and $1 \leq r \leq m_1 + 1$. A lexicographic order for multiple indices $I_{m_1+1} = (\beta_1\beta_2 \cdots \beta_{m_1}\beta_{m_1+1})$ is introduced, using

$$\mathcal{K}(I_{m_1+1}) = 1 + \sum_{r=2}^{m_1} 2^{m_2(m_1-r)}(\beta_r - 1). \quad (51)$$

Now, $A^{(k)}_{\hat{x};m_1,m_2,m_3;\alpha}$ can be represented by

$$A_{\hat{x};2\times m_2\times m_3;\beta_1\beta_2}A_{\hat{x};2\times m_2\times m_3;\beta_2\beta_3} \cdots A_{\hat{x};2\times m_2\times m_3;\beta_{m_1}\beta_{m_1+1}}, \quad (52)$$

where

$$\alpha = \alpha(\beta_1, \beta_{m_1+1}) = 2^{m_2}(\beta_1 - 1) + \beta_{m_1+1}$$

and

$$k = \mathcal{K}(I_{m_1+1})$$

as in Eq. (51). Accordingly, $\mathbb{A}^{m_1}_{\hat{x};2\times m_2\times m_3}$ can be expressed as

$$[A_{\hat{x};m_1,m_2,m_3;\alpha}]_{2^{m_2}\times 2^{m_2}}, \quad (53)$$

where $1 \leq \alpha \leq 2^{2m_2}$ and

$$A_{\hat{x};m_1,m_2,m_3;\alpha} = \sum_{k=1}^{2^{m_2(m_1-1)}} A^{(k)}_{\hat{x};m_1,m_2,m_3;\alpha}. \quad (54)$$

Moreover,

$$V_{\hat{x};m_1,m_2,m_3;\alpha} = (A^{(k)}_{\hat{x};m_1,m_2,m_3;\alpha})^t, \quad (55)$$

where $1 \leq k \leq 2^{m_2(m_1-1)}$, $V_{\hat{x};m_1,m_2,m_3;\alpha}$ is a $2^{m_2(m_1-1)}$ column vector that comprises all elementary patterns in $A_{\hat{x};m_1,m_2,m_3;\alpha}$. The ordering matrix $\mathbb{V}_{\hat{x};m_1,m_2,m_3}$ of $\mathbb{A}^{m_1}_{\hat{x};2\times m_2\times m_3}$ is now defined as

$$[V_{\hat{x};m_1,m_2,m_3;\alpha}]_{2^{m_2}\times 2^{m_2}}, \quad (56)$$

where $1 \leq \alpha \leq 2^{2m_2}$. The ordering matrix $\mathbb{V}_{\hat{x};m_1,m_2,m_3}$ allows the elementary patterns to be tracked during the reduction from $\mathbb{A}^{m_1}_{\hat{x};2\times m_2\times (m_3+1)}$ to $\mathbb{A}^{m_1}_{\hat{x};2\times m_2\times m_3}$. This careful book-keeping constitutes a systematic way to generate the admissible patterns, and as in Sec. 4.2, lower-bound estimates of spatial entropy.

This simplest example is considered first to illustrate this concept.

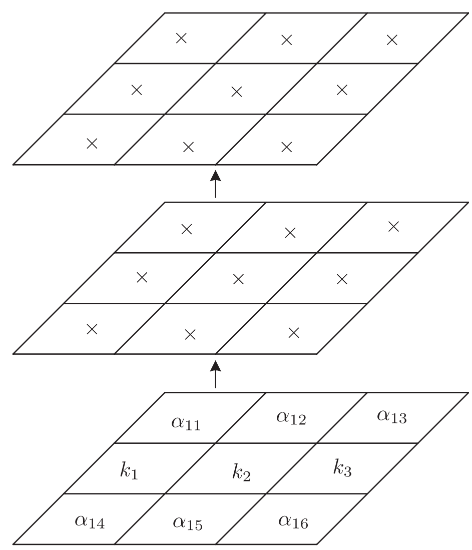
Example 4.1. For $m_1 = 2$, $m_2 = 3$, $m_3 = 3$, the following can be easily verified;

$$\mathbb{A}^2_{\hat{x};2\times 3\times 3} = [A_{\hat{x};2,3,3;\alpha_1}]_{2^3\times 2^3},$$

where $1 \leq \alpha_1 \leq 2^6$ and

$$A_{\hat{x};2,3,3;\alpha_1} = \sum_{k=1}^{2^3} A^{(k)}_{\hat{x};2,3,3;\alpha_1},$$

and for fixed α_1 and k the represented pattern of $A^{(k)}_{\hat{x};2,3,3;\alpha_1}$ are in the following form.



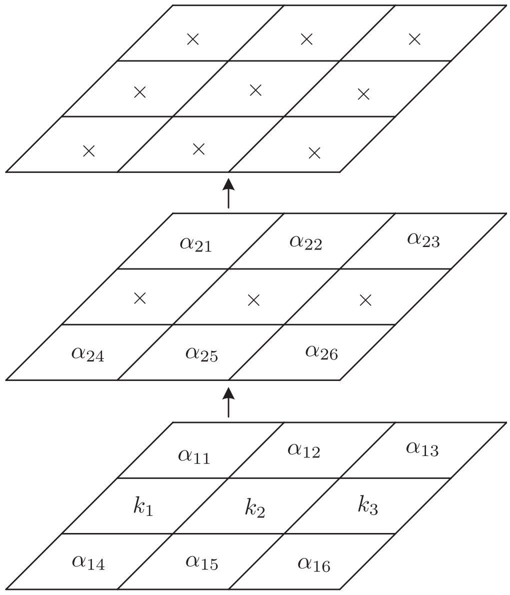
If the red symbol is defined equal to 1, and white symbol equals 0, then $\alpha_1 = 2^5\alpha_{11} + 2^4\alpha_{12} + 2^3\alpha_{13} + 2^2\alpha_{14} + 2\alpha_{15} + \alpha_{16} + 1$ and $k = 2^2k_1 + 2k_2 + k_3 + 1$. Hence

$$V_{\hat{x};2,3,3;\alpha_1} = (A^{(k)}_{\hat{x};2,3,3;\alpha_1})^t,$$

where $1 \leq k \leq 2^3$ and $1 \leq \alpha_1 \leq 2^6$. Define

$$V_{\hat{x};2,3,3;\alpha_1;\alpha_2} = (A^{(k)}_{\hat{x};2,3,3;\alpha_1;\alpha_2})^t,$$

where $1 \leq k \leq 2^3$ and $1 \leq \alpha_1, \alpha_2 \leq 2^6$ and the represented pattern of $A^{(k)}_{\hat{x};2,3,3;\alpha_1;\alpha_2}$ is



.

Therefore, for instance,

$$V_{\hat{x};2,3,3;1;1} = S_{\hat{x};m_3;2,3;11} V_{\hat{x};2,3,2;1},$$

and the represented patterns of $S_{\hat{x};m_3;2,3;11}$

$2^3 \times 2^3$.

The above derivation reveals that $V_{\hat{x};2,3,3,3;\alpha_1;\alpha_2}$ can be reduced to $V_{\hat{x};2,3,2;\alpha_2}$ by multiplication using connecting operator $S_{\hat{x};m_3;2,3;\alpha_1\alpha_2}$. This procedure can be extended to introduce the connecting operator $\mathbb{S}_{\hat{x};m_3;m_1m_2} = [S_{\hat{x};m_3;m_1m_2;\alpha_1\alpha_2}{}^4]$, where $1 \le \alpha_1, \alpha_2 \le 2^{2m_2}$, for all $m_1 \ge 2$, $m_2 \ge 2$.

Definition 4.2. For $m_1 \ge 2$, $m_2 \ge 2$, define

$$(\mathbb{C}_{\hat{x};m_3;m_1m_2})_{2^{2m_2}\times 2^{2m_2}} = (\mathbb{S}^{(r)}_{\hat{x};m_3;m_1m_2})_{2^{2m_2}\times 2^{2m_2}}, \tag{57}$$

where the row matrix $\mathbb{S}^{(r)}_{\hat{x};m_3;m_1m_2}$ of $\mathbb{S}_{\hat{x};m_3;m_1m_2}$ is defined in Eqs. (13) and (14). And

$$\begin{aligned} C_{\hat{x};m_3;m_1m_2;i_1i_2} &= [(A_{z;2\times m_2\times 2;i_1})_{2^{m_2}\times 2^{m_2}} \circ (\mathbb{A}_{z;(m_1-1)\times m_2\times 2})_{2^{m_2}\times 2^{m_2}}]_{2^{(m_1-1)m_2}\times 2^{(m_1-1)m_2}} \\ &\quad \circ (E_{2^{(m_1-2)m_2}} \otimes ((\mathbb{A}^{(r)}_{z;2\times m_2\times 2})^{(c)}_{;i_2})_{2^{m_2}\times 2^{m_2}})_{2^{(m_1-1)m_2}\times 2^{(m_1-1)m_2}} \end{aligned} \tag{58}$$

where $(\mathbb{A}^{(r)}_{z;2\times m_2\times 2})^{(c)}_{;i_2}$ is the i_2th block of the matrix $(\mathbb{A}^{(r)}_{z;2\times m_2\times 2})^{(c)}$, $(\mathbb{A}^{(r)}_{z;2\times m_2\times 2})^{(c)}$ is the column matrix of $\mathbb{A}^{(r)}_{z;2\times m_2\times 2}$, $\mathbb{A}^{(r)}_{z;2\times m_2\times 2}$ is the row matrix of $\mathbb{A}_{z;2\times m_2\times 2}$ and E_k is the $2^k \times 2^k$ matrix with 1 as its entries.

Remark 4.3. By a similar method, the following connecting operators can also be defined.

$$\begin{aligned} C_{x;m_2;m_1m_3;i_1i_2} &= [(A_{y;2\times 2\times m_3;i_1})_{2^{m_3}\times 2^{m_3}} \circ (\mathbb{A}_{y;(m_1-1)\times 2\times m_3})_{2^{m_3}\times 2^{m_3}}]_{2^{(m_1-1)m_3}\times 2^{(m_1-1)m_3}} \\ &\quad \circ (E_{2^{(m_1-2)m_3}} \otimes ((\mathbb{A}^{(r)}_{y;2\times 2\times m_3})^{(c)}_{;i_2})_{2^{m_3}\times 2^{m_3}})_{2^{(m_1-1)m_3}\times 2^{(m_1-1)m_3}} \\ C_{\hat{y};m_3;m_1m_2;i_1i_2} &= [(A_{\hat{z};m_1\times 2\times 2;i_1})_{2^{m_1}\times 2^{m_1}} \circ (\mathbb{A}_{\hat{z};m_1\times(m_2-1)\times 2})_{2^{m_1}\times 2^{m_1}}]_{2^{(m_2-1)m_1}\times 2^{(m_2-1)m_1}} \\ &\quad \circ (E_{2^{(m_2-2)m_1}} \otimes ((\mathbb{A}^{(r)}_{\hat{z};m_1\times 2\times 2})^{(c)}_{;i_2})_{2^{m_1}\times 2^{m_1}})_{2^{(m_2-1)m_1}\times 2^{(m_2-1)m_1}} \\ C_{y;m_1;m_2m_3;i_1i_2} &= [(A_{x;2\times 2\times m_3;i_1})_{2^{m_3}\times 2^{m_3}} \circ (\mathbb{A}_{x;2\times(m_2-1)\times m_3})_{2^{m_3}\times 2^{m_3}}]_{2^{(m_2-1)m_3}\times 2^{(m_2-1)m_3}} \\ &\quad \circ (E_{2^{(m_2-2)m_3}} \otimes ((\mathbb{A}^{(r)}_{x;2\times 2\times m_3})^{(c)}_{;i_2})_{2^{m_3}\times 2^{m_3}})_{2^{(m_2-1)m_3}\times 2^{(m_2-1)m_3}} \\ C_{\hat{z};m_2;m_1m_3;i_1i_2} &= [(A_{\hat{y};m_1\times 2\times 2;i_1})_{2^{m_1}\times 2^{m_1}} \circ (\mathbb{A}_{\hat{y};m_1\times 2\times(m_3-1)})_{2^{m_1}\times 2^{m_1}}]_{2^{(m_3-1)m_1}\times 2^{(m_3-1)m_1}} \\ &\quad \circ (E_{2^{(m_3-2)m_1}} \otimes ((\mathbb{A}^{(r)}_{\hat{y};m_1\times 2\times 2})^{(c)}_{;i_2})_{2^{m_1}\times 2^{m_1}})_{2^{(m_3-1)m_1}\times 2^{(m_3-1)m_1}} \\ C_{z;m_1;m_2m_3;i_1i_2} &= [(A_{\hat{x};2\times m_2\times 2;i_1})_{2^{m_2}\times 2^{m_2}} \circ (\mathbb{A}_{\hat{x};2\times m_2\times(m_3-1)})_{2^{m_2}\times 2^{m_2}}]_{2^{(m_3-1)m_2}\times 2^{(m_3-1)m_2}} \\ &\quad \circ (E_{2^{(m_3-2)m_2}} \otimes ((\mathbb{A}^{(r)}_{\hat{x};2\times m_2\times 2})^{(c)}_{;i_2})_{2^{m_2}\times 2^{m_2}})_{2^{(m_3-1)m_2}\times 2^{(m_3-1)m_2}} \end{aligned}$$

Theorem 4.4. *For any $m_2 \ge 2$, $m_3 \ge 2$ and $1 \le i_1, i_2 \le 2^{2m_2}$,*

$$C_{\hat{x};m_3;(m_1+1)m_2;i_1i_2} = [a_{\hat{x};2\times m_2\times 2;i_1i}C_{\hat{x};m_3;m_1m_2;ii_2}], \tag{59}$$

where $1 \le i \le 2^{2m_2}$.

Proof. By Theorem 3.5 and Remark 3.6,

$$\mathbb{A}_{z;m_1\times m_2\times 2} = [A_{z;2\times m_2\times 2;i_1} \circ \mathbb{A}_{z;(m_1-1)\times m_2\times 2}],$$

where $1 \le i_1 \le 2^{2m_2}$. Hence, by

$$\begin{aligned} C_{\hat{x};m_3;(m_1+1)m_2;i_1i_2} &= [(A_{z;2\times m_2\times 2;i_1}) \circ \mathbb{A}_{z;m_1\times m_2\times 2}] \circ [E_{2^{(m_1-1)m_2}} \otimes (A^{(r)}_{z;2\times m_2\times 2})^{(c)}_{;i_2}] \\ &= [a_{\hat{x};2\times m_2\times 2;i_1i}(A_{z;2\times m_2\times 2;i} \circ \mathbb{A}_{z;(m_1-1)\times m_2\times 2})] \circ [E_{2^{m_2}} \otimes (E_{2^{(m_1-2)m_2}} \otimes (A^{(r)}_{z;2\times m_2\times 2})^{(c)}_{;i_2})] \\ &= [a_{\hat{x};2\times m_2\times 2;i_1i}C_{\hat{x};m_3;m_1m_2;ii_2}]_{2^{m_2}\times 2^{m_2}} \end{aligned}$$

where $1 \le i \le 2^{2m_2}$. The proof is complete. ■

Notably, Eq. (59) implies $C_{\hat{x};m_3;m_1m_2;ij}$ is

$$a_{\hat{x};2\times m_2\times 2;i_1i_2}a_{\hat{x};2\times m_2\times 2;i_2i_3}\cdots a_{\hat{x};2\times m_2\times 2;i_{m_1}i_{m_1+1}}$$

[4] Use $S_{\hat{x};m_3;m_1m_2;\alpha_1\alpha_2}$ to substitute $S_{\hat{x};m_3;m_1m_2;\alpha_1,\alpha_2}$ for simplicity afterward.

with $i_1 = i$ and $i_{m_1+1} = j$. $C_{\hat{x};m_3;m_1m_2;ij}$ comprises all paths of length m_1+1, that start at i and end at j. Indeed, the entries of $\mathbb{C}_{\hat{x};m_3;m_1m_2}$ and $\mathbb{A}_{z;(m_1+1)\times m_2\times 2}$ are the same. However, the arrangements differ.

Substituting m_3 for m_3+1 into Eq. (52) and using Eq. (39), $A^{(k)}_{\hat{x};m_1,m_2,m_3+1;\alpha}$ could be represented by

$$\begin{aligned}&A_{\hat{x};2\times m_2\times(m_3+1);\beta_1\beta_2}A_{\hat{x};2\times m_2\times(m_3+1);\beta_2\beta_3}\\&\quad\cdots A_{\hat{x};2\times m_2\times(m_3+1);\beta_{m_1}\beta_{m_1+1}}\\&\quad=\prod_{j=1}^{m_1}[a_{\hat{x};2\times m_2\times 2;\alpha_j\hat{\alpha}}A_{\hat{x};2\times m_2\times m_3;\hat{\beta}_1\hat{\beta}_2}]_{2^{m_2}\times 2^{m_2}},\end{aligned}\tag{60}$$

where $1\le\hat{\beta}_1,\hat{\beta}_2\le 2^{m_2}$ and $\alpha_j=\alpha(\beta_j,\beta_{j+1})$ and $\hat{\alpha}=\alpha(\hat{\beta}_1,\hat{\beta}_2)$ for $1\le j\le m_1$.

After m_1 matrix multiplications have been performed as in Eq. (60),

$$A^{(k)}_{\hat{x};m_1,m_2,m_3+1;\alpha_1}=[A^{(k)}_{\hat{x};m_1,m_2,m_3+1;\alpha_1;\alpha_2}]_{2^{m_2}\times 2^{m_2}},\tag{61}$$

where $1\le\alpha_2\le 2^{2m_2}$ and $A^{(k)}_{\hat{x};m_1,m_2,m_3+1;\alpha_1;\alpha_2}$ can be represented by

$$\sum_{\ell=1}^{2^{m_2(m_1-1)}}K(\hat{x},m_1m_2;\alpha_1\alpha_2;k,\ell)A^{(\ell)}_{\hat{x};m_1,m_2,m_3;\alpha_2}\tag{62}$$

which is a linear combination of $A^{(\ell)}_{\hat{x};m_1,m_2,m_3;\alpha_2}$ with the coefficients $K(\hat{x},m_1m_2;\alpha_1\alpha_2;k,\ell)$ which are products of $a_{\hat{x};2\times m_2\times 2;\alpha_j\hat{\alpha}}$, $1\le j\le m_1$. $K(\hat{x},m_1m_2;\alpha_1\alpha_2;k,\ell)$ must be studied in more detail. Notably,

$$\mathbb{A}^{m_1}_{\hat{x};2\times m_2\times(m_3+1)}=[A_{\hat{x};m_1,m_2,m_3+1;\alpha_1}]_{2^{m_2}\times 2^{m_2}}\tag{63}$$

where $1\le\alpha_1\le 2^{2m_2}$,

$$A_{\hat{x};m_1,m_2,m_3+1;\alpha_1}=\sum_{k=1}^{2^{m_2(m_1-1)}}A^{(k)}_{\hat{x};m_1,m_2,m_3+1;\alpha_1}$$

and

$$\sum_{k=1}^{2^{m_2(m_1-1)}}A^{(k)}_{\hat{x};m_1,m_2,(m_3+1);\alpha_1}=\left[\sum_{k=1}^{2^{m_2(m_1-1)}}A^{(k)}_{\hat{x};m_1,m_2,(m_3+1);\alpha_1;\alpha_2}\right]_{2^{m_2}\times 2^{m_2}},$$

where $1\le\alpha_2\le 2^{2m_2}$. Now, $V_{\hat{x};m_1,m_2,m_3+1;\alpha_1;\alpha_2}$ is defined as

$$V_{\hat{x};m_1,m_2,m_3+1;\alpha_1;\alpha_2}=(A^{(k)}_{\hat{x};m_1,m_2,m_3+1;\alpha_1;\alpha_2})^t.\tag{64}$$

From Eqs. (62) and (64),

$$\begin{aligned}&V_{\hat{x};m_1,m_2,m_3+1;\alpha_1;\alpha_2}\\&\quad=\mathbb{K}(\hat{x},m_1m_2;\alpha_1\alpha_2)V_{\hat{x};m_1,m_2,m_3;\alpha_2}\end{aligned}\tag{65}$$

where

$$\mathbb{K}(\hat{x},m_1m_2;\alpha_1\alpha_2)=(K(\hat{x},m_1m_2;\alpha_1\alpha_2;k,\ell)),$$

$1\le k,\ell\le 2^{m_2(m_1-1)}$ is a $2^{m_2(m_1-1)}\times 2^{m_2(m_1-1)}$ matrix. Now

$$\mathbb{K}(\hat{x},m_1m_2;\alpha_1\alpha_2)=S_{\hat{x};m_3;m_1m_2;\alpha_1\alpha_2}$$

must be shown as follows.

Theorem 4.5. *For any $m_1\ge 2$, $m_2\ge 2$ and $m_3\ge 2$, let $S_{\hat{x};m_3;m_1m_2;\alpha_1\alpha_2}$ be given as in Eqs. (57) and (58). Then,*

$$\begin{aligned}&V_{\hat{x};m_1,m_2,m_3+1;\alpha_1;\alpha_2}\\&\quad=S_{\hat{x};m_3;m_1m_2;\alpha_1\alpha_2}V_{\hat{x};m_1,m_2,m_3;\alpha_2},\end{aligned}\tag{66}$$

or equivalently, the recursive formula

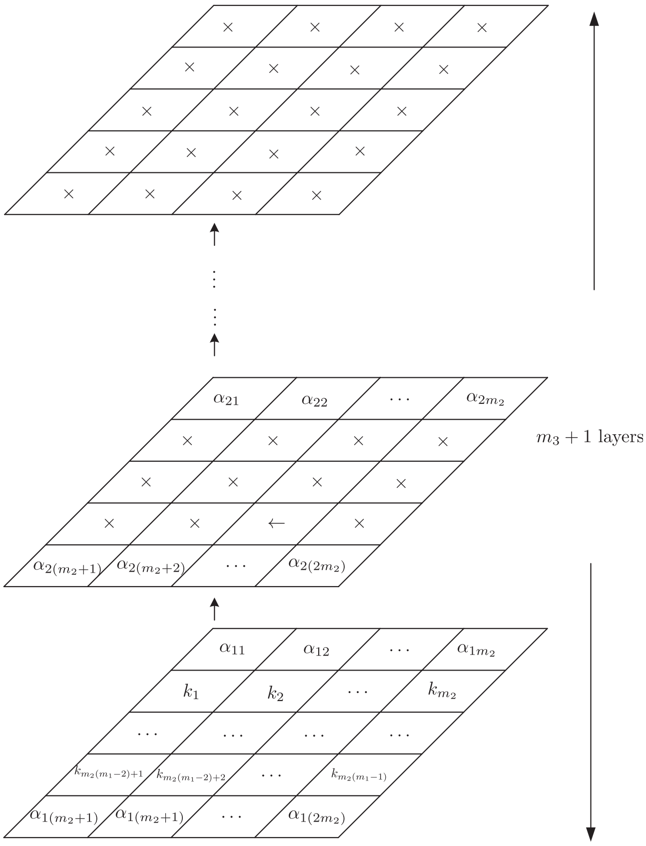
$$A^{(k)}_{\hat{x};m_1,m_2,(m_3+1);\alpha_1}=\left[\sum_{\ell=1}^{2^{m_2(m_1-1)}}(S_{\hat{x};m_3;m_1m_2;\alpha_1\alpha_2})_{k\ell}A^{(\ell)}_{\hat{x};m_1,m_2,m_3;\alpha_2}\right]_{2^{m_2}\times 2^{m_2}},\tag{67}$$

where $1\le\alpha_2\le 2^{2m_2}$. Moreover, for $m_3=1$,

$$A^{(k)}_{\hat{x};m_1,m_2,2;\alpha_1}=\left[\sum_{\ell=1}^{2^{m_2(m_1-1)}}(S_{\hat{x};m_3;m_1m_2;\alpha_1\alpha_2})_{k\ell}\right]_{2^{m_2}\times 2^{m_2}},\tag{68}$$

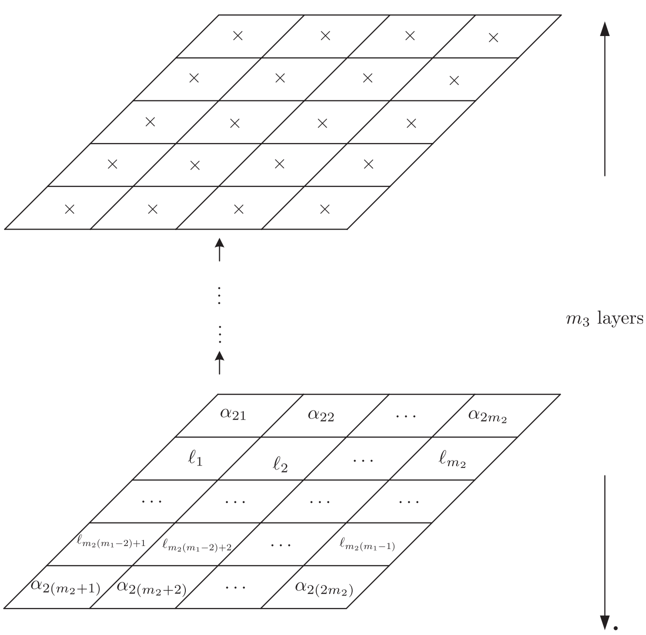
where $1\le\alpha_2\le 2^{2m_2}$ for any $1\le k\le 2^{m_2(m_1-1)}$ and $\alpha_1\in\{1,2,\ldots,2^{2m_2}\}$.

Proof. From Eq. (61), $A^{(k)}_{\hat{x};m_1,m_2,(m_3+1);\alpha_1;\alpha_2}$ can be represented as the pattern



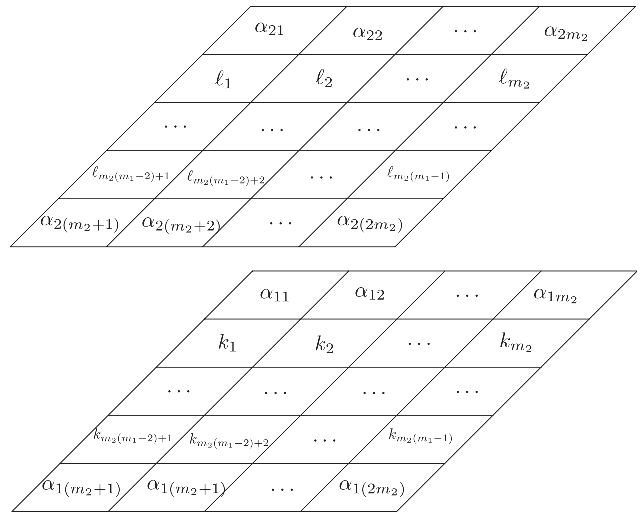
(69)

and $A^{(\ell)}_{\hat{x};m_1,m_2,m_3;\alpha_2}$ as the pattern,



(70)

From Definition 4.2, $S_{\hat{x};m_3;m_1m_2;\alpha_1\alpha_2}$ represents the following pattern



(71)

Therefore, Eq. (67) follows from Eqs. (69)–(71). Also, from Eq. (65), Eq. (66) follows.

Next, Eq. (68) follows simply from Eqs. (69) and (71). ∎

For any positive integer $p \geq 2$, applying Theorem 4.5 p times allows the elementary patterns of $\mathbb{A}^{m_1}_{\hat{x};2\times m_2\times(m_3+p)}$ to be expressed as products of a sequence of $S_{\hat{x};m_3;m_1m_2;\alpha_i\alpha_{i+1}}$ and the elementary patterns in $\mathbb{A}^{m_1}_{\hat{x};2\times m_2\times m_3}$. The elementary pattern in $\mathbb{A}^{m_1}_{\hat{x};2\times m_2\times(m_3+p)}$ is first considered. For any $p \geq 2$ and $1 \leq q \leq p-1$, define

$$A^{(k)}_{\hat{x};m_1,m_2,m_3+p;\alpha_1;\alpha_2;\ldots;\alpha_q} = [A^{(k)}_{\hat{x};m_1,m_2,m_3+p;\alpha_1;\alpha_2;\ldots;\alpha_q;\alpha_{q+1}}]_{2^{m_2}\times 2^{m_2}},$$

where $1 \leq \alpha_{q+1} \leq 2^{2m_2}$. Then

$$A^{(k)}_{\hat{x};m_1,m_2,m_3+p;\alpha_1;\alpha_2;\ldots;\alpha_{p+1}}$$

can be represented as

$$\sum_{\ell_2=1}^{2^{m_2(m_1-1)}} \sum_{\ell_3=1}^{2^{m_2(m_1-1)}} \cdots \sum_{\ell_{p+1}=1}^{2^{m_2(m_1-1)}} \times \left(\prod_{i=2}^{p+1} K(\hat{x}; m_1m_2; \alpha_{i-1}\alpha_i; \ell_{i-1}, \ell_i) \right) \times A^{(\ell_{p+1})}_{\hat{x};m_1;m_2;m_3;\alpha_{p+1}} \quad (72)$$

where and $\ell_1 = k$ can be easily verified.

Hence, for any $p \geq 2$, Eq. (63) can be generalized for $\mathbb{A}^{m_1}_{\hat{x};2\times m_2\times(m_3+p)}$ as a $(2^{m_2})^{p+1} \times (2^{m_2})^{p+1}$

matrix

$$\mathbb{A}^{m_1}_{\hat{x};2\times m_2\times(m_3+p)} = [A_{\hat{x};m_1,m_2,(m_3+p);\alpha_1;\alpha_2;\ldots;\alpha_{p+1}}], \tag{73}$$

where

$$A_{\hat{x};m_1,m_2,(m_3+p);\alpha_1;\alpha_2;\ldots;\alpha_{p+1}} = \sum_{k=1}^{2^{(m_1-1)m_2}} A^{(k)}_{\hat{x};m_1,m_2,(m_3+1);\alpha_1;\alpha_2;\ldots;\alpha_{p+1}}.$$

In particular, if $\alpha_1, \alpha_2, \ldots, \alpha_{p+1} \in \{2^{m_2}(s-1)+s | 1 \le s \le 2^{m_2}\}$ then $A_{\hat{x};m_1,m_2,(m_3+p);\alpha_1;\alpha_2;\ldots;\alpha_{p+1}}$ lies on the diagonal of $\mathbb{A}^{m_1}_{\hat{x};2\times m_2\times(m_3+p)}$ in Eq. (73). Now, define

$$V_{\hat{x};m_1,m_2,m_3+p;\alpha_1;\alpha_2;\ldots;\alpha_{p+1}} = (A^{(k)}_{m_1,m_2,m_3+p;\alpha_1;\alpha_2;\ldots;\alpha_{p+1}})^t.$$

Therefore, Theorem 4.5 can be generalized to the following Theorem.

Theorem 4.6. *For any* $m_1 \ge 2$, $m_2 \ge 2$, $m_3 \ge 2$ *and* $p \ge 1$, $V_{\hat{x};m_1,m_2,m_3+p;\alpha_1;\alpha_2;\ldots;\alpha_{p+1}}$ *could be represented as*

$$S_{\hat{x};m_3;m_1m_2;\alpha_1\alpha_2} S_{\hat{x};m_3;m_1m_2;\alpha_2\alpha_3} \cdots S_{\hat{x};m_3;m_1m_2;\alpha_p\alpha_{p+1}} V_{\hat{x};m_1,m_2,m_3;\alpha_{p+1}}$$

where $1 \le \alpha_i \le 2^{2m_2}$ *and* $1 \le i \le p+1$.

Proof. From Eqs. (72), (65) and (67),

$$\begin{aligned}
&A^{(k)}_{\hat{x};m_1,m_2,m_3+p;\alpha_1;\alpha_2;\ldots;\alpha_{p+1}} \\
&\quad = \sum_{\ell_2=1}^{2^{m_2(m_1-1)}} \sum_{\ell_3=1}^{2^{m_2(m_1-1)}} \cdots \sum_{\ell_{p+1}=1}^{2^{m_2(m_1-1)}} \left(\prod_{i=2}^{p+1} K(\hat{x}; m_1m_2; \alpha_{i-1}\alpha_i; \ell_{i-1}, \ell_i) \right) A^{(\ell_{p+1})}_{\hat{x};m_1,m_2,m_3;\alpha_{p+1}} \\
&\quad = \sum_{\ell_2=1}^{2^{m_2(m_1-1)}} \sum_{\ell_3=1}^{2^{m_2(m_1-1)}} \cdots \sum_{\ell_{p+1}=1}^{2^{m_2(m_1-1)}} \left(\prod_{i=2}^{p+1} (S_{\hat{x};m_3;m_1m_2;\alpha_{i-1}\alpha_i})_{\ell_{i-1}\ell_i} \right) A^{(\ell_{p+1})}_{\hat{x};m_1,m_2,m_3;\alpha_{p+1}} \\
&\quad = \sum_{\ell_2=1}^{2^{m_2(m_1-1)}} \sum_{\ell_3=1}^{2^{m_2(m_1-1)}} \cdots \sum_{\ell_{p+1}=1}^{2^{m_2(m_1-1)}} ((S_{\hat{x};m_3;m_1m_2;\alpha_1\alpha_2})_{\ell_1\ell_2} (S_{\hat{x};m_3;m_1m_2;\alpha_2\alpha_3})_{\ell_2\ell_3} \\
&\qquad \cdots (S_{\hat{x};m_3;m_1m_2;\alpha_p\alpha_{p+1}})_{\ell_p\ell_{p+1}}) A^{(\ell_{p+1})}_{\hat{x};m_1,m_2,m_3;\alpha_{p+1}} \\
&\quad = \sum_{\ell_{p+1}=1}^{2^{m_2(m_1-1)}} (S_{\hat{x};m_3;m_1m_2;\alpha_1\alpha_2} S_{\hat{x};m_3;m_1m_2;\alpha_2\alpha_3} \cdots S_{\hat{x};m_3;m_1m_2;\alpha_p\alpha_{p+1}})_{k\ell_{p+1}} A^{(\ell_{p+1})}_{\hat{x};m_1,m_2,m_3;\alpha_{p+1}}.
\end{aligned}$$

The proof is complete. ■

4.2. *Lower bound of entropy*

In this subsection, the connecting operator $\mathbb{C}_{\hat{x};m_3;m_1m_2}$ is adopted to estimate the lower bound of entropy and in particular, to confirm that is positive. The following notation is used.

Definition 4.7. Let $V = (V_1, \ldots, V_M)^t$, where V_k are $N \times N$ matrices. Define the sum over V_k as

$$|V| = \sum_{k=1}^{N} V_k. \tag{74}$$

If $\mathbb{M} = [M_{ij}]$ is a $M \times M$ matrix, then

$$|\mathbb{M}V| = \sum_{i=1}^{M} \sum_{j=1}^{M} M_{ij} V_j$$

Notably, (74) implies

$$\begin{aligned}
|V_{\hat{x};m_1,m_2,m_3;\alpha}| &= \sum_{k=1}^{2^{(m_1-1)m_2}} A^{(k)}_{\hat{x};m_1,m_2,m_3;\alpha} \\
&= A_{\hat{x};m_1,m_2,m_3;\alpha}.
\end{aligned}$$

As is typical, the set of all matrices with the same order can be partially ordered.

Definition 4.8. Let $\mathbb{M} = [M_{ij}]$ and $\mathbb{N} = [N_{ij}]$ be two $M \times M$ matrices; $\mathbb{M} \geq \mathbb{N}$ if $M_{ij} \geq N_{ij}$ for all $1 \leq i, j \leq M$.

Notably, if $\mathbb{A}_{x;2\times 2\times 2} \geq \mathbb{A}'_{x;2\times 2\times 2}$, then $\mathbb{A}_{\hat{x};2\times m_2\times m_3} \geq \mathbb{A}'_{\hat{x};2\times m_2\times m_3}$ for all m_2, $m_3 \geq 2$. Therefore, $h(\mathbb{A}_{x;2\times 2\times 2}) \geq h(\mathbb{A}'_{x;2\times 2\times 2})$. Hence, the spatial entropy as a function of $\mathbb{A}_{x;2\times 2\times 2}$ is monotonic with respect to the partial order $\geq$.

Definition 4.9. A $P+1$ multiple index

$$\mathcal{A}_P \equiv (\alpha_1\alpha_2 \cdots \alpha_P\alpha_{P+1}) \tag{75}$$

is called a periodic cycle if

$$\alpha_{P+1} = \alpha_1, \tag{76}$$

where $1 \leq \alpha_i \leq 2^{2m_2}$ and $1 \leq i \leq P+1$. It is called diagonal cycle if Eq. (76) holds and

$$\alpha_i \in \{2^{m_2}(s-1) + s | 1 \leq s \leq 2^{m_2}\}$$

for each $1 \leq i \leq P+1$. For a diagonal cycle Eq. (75)

$$\bar{\alpha_P} = \alpha_1; \alpha_2; \cdots; \alpha_P$$

and

$$\bar{\alpha_P}^n = \bar{\alpha_P}; \bar{\alpha_P}; \cdots; \bar{\alpha_P}. \quad (n\text{-times})$$

First, prove the following lemma.

Lemma 4.10. *Let $m_1 \geq 2$, $m_2 \geq 2$, $P \geq 1$, $\mathcal{A}_P$ be a diagonal cycle. Then, for any $m_3 \geq 1$,*

$$\begin{aligned}\rho(\mathbb{A}^{m_1}_{\hat{x};2\times m_2\times(m_3P+2)}) &\geq \rho(|(S_{\hat{x};m_3;m_1m_2;\alpha_1\alpha_2}S_{\hat{x};m_3;m_1m_2;\alpha_2\alpha_3}\\ &\quad \cdots S_{\hat{x};m_3;m_1m_2;\alpha_P\alpha_{P+1}})^{m_3}V_{\hat{x};m_1,m_2,2;\alpha_1}|).\end{aligned} \tag{77}$$

Proof. Since $\mathcal{A}_P$ is a periodic cycle, Theorem 4.6 implies

$$\begin{aligned}V_{\hat{x};m_1,m_2,m_3P+2;\bar{\alpha_P}^{m_3};\alpha_1} &= (S_{\hat{x};m_3;m_1m_2;\alpha_1\alpha_2}S_{\hat{x};m_3;m_1m_2;\alpha_2\alpha_3}\\ &\quad \cdots S_{\hat{x};m_3;m_1m_2;\alpha_P\alpha_{P+1}})^{m_3}V_{\hat{x};m_1,m_2,2;\alpha_1}.\end{aligned} \tag{78}$$

Furthermore, $\mathcal{A}_P$ is diagonal and $|V_{\hat{x};m_1,m_2,m_3P+2;\bar{\alpha_P}^{m_3};\alpha_1}| = A_{\hat{x};m_1,m_2,m_3P+2;\bar{\alpha_P}^{m_3};\alpha_1}$ lies in the diagonal part of Eq. (73), with $m_3 + p = m_3P + 2$. Accordingly,

$$\rho(\mathbb{A}^{m_1}_{\hat{x};m_1,m_2,m_3P+2}) \geq \rho(|V_{\hat{x};m_1,m_2,m_3P+2;\bar{\alpha_P}^{m_3};\alpha_1}|). \tag{79}$$

Therefore, Eq. (77) follows from Eqs. (78) and (79). The proof is complete. ■

The following lemma is useful in evaluating maximum eigenvalue of Eq. (77).

Lemma 4.11. *For any $m_1 \geq 2$, $m_2 \geq 2$, $1 \leq k \leq 2^{(m_1-1)m_2}$ and $\alpha_1 \in \{(s-1)2^{m_2} + s | 1 \leq s \leq 2^{m_2}\}$, if*

$$\mathrm{tr}(A^{(k)}_{\hat{x};m_1,m_2,2;\alpha_1}) = 0,$$

then for all $1 \leq \ell \leq 2^{(m_1-1)m_2}$,

$$(S_{\hat{x};m_3;m_1m_2;\alpha_1\alpha_2})_{k\ell} = 0, \tag{80}$$

for all $\alpha_2 \in \{(s-1)2^{m_2} + s | 1 \leq s \leq 2^{m_2}\}$, such that the kth rows of matrices $S_{\hat{x};m_3;m_1m_2;\alpha_1\alpha_2}$ are zeros. For any diagonal cycle $\mathcal{A}_P$, let $U = (u_1u_2 \cdots u_{2^{m_2(m_1-1)}})$ be an eigenvector of $S_{\hat{x};m_3;m_1m_2;\alpha_1\alpha_2}$ $S_{\hat{x};m_3;m_1m_2;\alpha_2\alpha_3} \cdots S_{\hat{x};m_3;m_1m_2;\alpha_P\alpha_1}$. If $u_k \neq 0$ for some $1 \leq k \leq 2^{(m_1-1)m_2}$, then $\mathrm{tr}(A^{(k)}_{\hat{x};m_1,m_2,2;\alpha_1}) > 0$.

Proof. Since $A^{(k)}_{\hat{x};m_1,m_2,2;\alpha_1}$ can be expressed as Eq. (68). $tr(A^{(k)}_{\hat{x};m_1,m_2,2;\alpha_1}) = 0$ if and only if Eq. (80) holds for all $1 \leq \ell \leq 2^{(m_1-1)m_2}$. The second part of the Lemma 4.11 follows easily from the first part. The proof is complete. ■

By Lemmas 4.10 and 4.11, the lower bound of entropy can be determined as follows.

Theorem 4.12. *Let $\alpha_1\alpha_2 \cdots \alpha_P\alpha_1$ be a diagonal cycle. Then, for any $m_1 \geq 2$, $m_2 \geq 2$,*

$$h(\mathbb{A}_{x;2\times 2\times 2}) \geq \lim_{m_2\to\infty} \frac{1}{m_1m_2P} \log \rho(S_{\hat{x};m_3;m_1m_2;\alpha_1\alpha_2}S_{\hat{x};m_3;m_1m_2;\alpha_2\alpha_3} \cdots S_{\hat{x};m_3;m_1m_2;\alpha_P\alpha_1}). \tag{81}$$

Proof. First, by the methods used to prove Lemmas 2.10 and 2.11 and Theorem 2.12 in [Ban *et al.*, 2007],

$$\begin{aligned}&\limsup_{m_3\to\infty} \frac{1}{m_3}(\log \rho(|(S_{\hat{x};m_3;m_1m_2;\alpha_1\alpha_2}S_{\hat{x};m_3;m_1m_2;\alpha_2\alpha_3} \cdots S_{\hat{x};m_3;m_1m_2;\alpha_P\alpha_1})^{m_3}V_{\hat{x};m_1,m_2,2;\alpha_1}|))\\ &\quad = \log \rho(S_{\hat{x};m_3;m_1m_2;\alpha_1\alpha_2}S_{\hat{x};m_3;m_1m_2;\alpha_2\alpha_3} \cdots S_{\hat{x};m_3;m_1m_2;\alpha_P\alpha_1})\end{aligned} \tag{82}$$

is obtained. The detailed proofs are omitted here for brevity. Now,

$$\begin{aligned}h(\mathbb{A}_{x;2\times 2\times 2}) \geq \lim_{m_2\to\infty} \frac{1}{m_1m_2P} \limsup_{m_3\to\infty} \frac{1}{m_3}(\log \rho(|(S_{\hat{x};m_3;m_1m_2;\alpha_1\alpha_2}S_{\hat{x};m_3;m_1m_2;\alpha_2\alpha_3}\\ \cdots S_{\hat{x};m_3;m_1m_2;\alpha_P\alpha_1})^{m_3}V_{\hat{x};m_1,m_2,2;\alpha_1}|))\end{aligned}$$

is established. Indeed, from Eqs. (40) and (77),

$$
\begin{aligned}
h(\mathbb{A}_{x;2\times2\times2}) &= \lim_{m_2 m_3\to\infty} \frac{1}{(m_3P+2)m_2} \log\rho(\mathbb{A}_{\hat{x};2\times m_2\times(m_3P+2)}) \\
&= \lim_{m_2 m_3\to\infty} \frac{1}{m_1(m_3P+2)m_2} \log\rho(\mathbb{A}^{m_1}_{\hat{x};2\times m_2\times(m_3P+2)}) \\
&\geq \lim_{m_2\to\infty} \frac{1}{m_1 m_2 P} \limsup_{m_3\to\infty} \frac{1}{m_3} (\log\rho(|(S_{\hat{x};m_3;m_1m_2;\alpha_1\alpha_2} S_{\hat{x};m_3;m_1m_2;\alpha_2\alpha_3} \\
&\quad \cdots S_{\hat{x};m_3;m_1m_2;\alpha_P\alpha_1})^{m_3} V_{\hat{x};m_1,m_2,2;\alpha_1}|)).
\end{aligned}
$$

Apply Eq. (82) which completes the proof. ■

Remark 4.13. By the similar method, the following lower bounds of entropy can also be estimated.

$$h(\mathbb{A}_{x;2\times2\times2}) \geq \lim_{m_3\to\infty} \frac{1}{m_1 m_3 P} \log\rho(S_{x;m_2;m_1m_3;\alpha_1\alpha_2} S_{x;m_2;m_1m_3;\alpha_2\alpha_3} \cdots S_{x;m_2;m_1m_3;\alpha_P\alpha_1}).$$

$$h(\mathbb{A}_{x;2\times2\times2}) \geq \lim_{m_1\to\infty} \frac{1}{m_1 m_2 P} \log\rho(S_{\hat{y};m_3;m_1m_2;\alpha_1\alpha_2} S_{\hat{y};m_3;m_1m_2;\alpha_2\alpha_3} \cdots S_{\hat{y};m_3;m_1m_2;\alpha_P\alpha_1}).$$

$$h(\mathbb{A}_{x;2\times2\times2}) \geq \lim_{m_3\to\infty} \frac{1}{m_2 m_3 P} \log\rho(S_{y;m_1;m_2m_3;\alpha_1\alpha_2} S_{y;m_1;m_2m_3;\alpha_2\alpha_3} \cdots S_{\hat{x};m_1;m_2m_3;\alpha_P\alpha_1}).$$

$$h(\mathbb{A}_{x;2\times2\times2}) \geq \lim_{m_1\to\infty} \frac{1}{m_1 m_3 P} \log\rho(S_{\hat{z};m_2;m_1m_3;\alpha_1\alpha_2} S_{\hat{z};m_2;m_1m_3;\alpha_2\alpha_3} \cdots S_{\hat{z};m_2;m_1m_3;\alpha_P\alpha_1}).$$

$$h(\mathbb{A}_{x;2\times2\times2}) \geq \lim_{m_2\to\infty} \frac{1}{m_2 m_3 P} \log\rho(S_{z;m_1;m_2m_3;\alpha_1\alpha_2} S_{z;m_1;m_2m_3;\alpha_2\alpha_3} \cdots S_{z;m_1;m_2m_3;\alpha_P\alpha_1}).$$

Remark 4.14. The results in last three sections can be generated to p-symbols on $\mathbf{Z}_{2\ell\times2\ell\times2\ell}$ such as in two-dimensional case [Ban & Lin, 2005] and [Ban *et al.*, 2007] and the details are omitted here for brevity.

5. Applications to 3DCNN

This section elucidates an interesting model in 3DCNN of the application of the method. The method is elucidated by considering $a_{0,0,0} = a$, $a_{1,0,0} = a_x$, $a_{0,1,0} = a_y$ and $a_{0,0,1} = a_z$, which are nonzero; in other cases, $a_{\alpha,\beta,\gamma}$ and $b_{\alpha,\beta,\gamma}$ are zero. Then, the 3DCNN is of the form as in Eq. (6)

$$
\begin{aligned}
\frac{du_{i,j,k}}{dt} &= -u_{i,j,k} + w + af(u_{i,j,k}) + a_x f(u_{i+1,j,k}) \\
&\quad + a_y f(u_{i,j+1,k}) + a_z f(u_{i,j,k+1}).
\end{aligned}
$$

The stationary solution to Eq. (6) satisfies

$$
\begin{aligned}
u_{i,j,k} &= w + av_{i,j,k} + a_x v_{i+1,j,k} + a_y v_{i,j+1,k} \\
&\quad + a_z v_{i,j,k+1},
\end{aligned}
$$

for $(i,j,k) \in \mathbf{Z}^3$ as in Eq. (7).

Firstly, consider the mosaic solution $u = (u_{i,j,k})$ to Eq. (7). If $u_{i,j,k} \geq 1$, i.e. $v_{i,j,k} = 1$, then

$$
\begin{aligned}
&(a-1) + w + a_x v_{i+1,j,k} + a_y v_{i,j+1,k} \\
&\quad + a_z v_{i,j,k+1} \geq 0. \qquad (83)
\end{aligned}
$$

If $u_{i,j,k} \leq -1$, i.e. $v_{i,j,k} = -1$, then

$$
\begin{aligned}
&(a-1) - w - (a_x v_{i+1,j,k} + a_y v_{i,j+1,k} + a_z v_{i,j,k+1}) \\
&\quad \geq 0. \qquad (84)
\end{aligned}
$$

Equation (7) has five parameters w, a, a_x, a_y and a_z. Three procedures are adopted to partition these parameters:

Procedure (I). The parameters a_x, a_y and a_z are initially expressed into three-dimensional coordinates, to solve Eqs. (83) and (84), as in Fig. 3.

Clearly 2^3 octants (I)–(VIII) exist in (a_x, a_y, a_z) three-dimensional coordinates.

Procedure (II). In each octant are 3! relations

$$
\begin{aligned}
&\text{(i)} \ : |a_x| > |a_y| > |a_z|, \\
&\text{(ii)} \ : |a_x| > |a_z| > |a_y|, \\
&\text{(iii)} : |a_y| > |a_x| > |a_z|, \\
&\text{(iv)} : |a_y| > |a_z| > |a_x|, \\
&\text{(v)} \ : |a_z| > |a_x| > |a_y|, \\
&\text{(vi)} : |a_z| > |a_y| > |a_x|.
\end{aligned}
\qquad (85)
$$

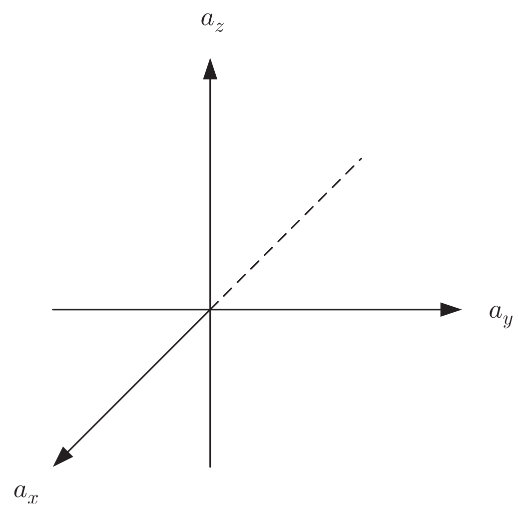


Fig. 3. Primary partition of (a_x, a_y, a_z).

Procedure (III). Each relation, denoted by $|a_1| > |a_2| > |a_3|$, two situations apply

$$\begin{aligned} &(1)\ |a_1| > |a_2| + |a_3| \\ &(2)\ |a_1| < |a_2| + |a_3|. \end{aligned} \tag{86}$$

However, in the (a, w)-planes, two sets of 2^3 straight lines are important. The first set is

$$\begin{aligned} \ell_r^+ : (a-1) + w + a_x v_{i+1,j,k} + a_y v_{i,j+1,k} \\ + a_z v_{i,j,k+1} = 0, \end{aligned}$$

which is related to Eq. (83). The second set is

$$\begin{aligned} \ell_r^- : (a-1) - w - (a_x v_{i+1,j,k} + a_y v_{i,j+1,k} \\ + a_z v_{i,j,k+1}) = 0, \end{aligned}$$

which is related to Eq. (84), where $v_{i+1,j,k}, v_{i,j+1,k}, v_{i,j,k+1} \in \{-1, 1\}$ and $1 \leq r \leq 8$. When (a_x, a_y, a_z) lines in the open region (I)–(VIII), (i)–(vi) and (1)–(2) as in Fig. 3, Eqs. (85) and (86) are used to partition the $(w, a-1)$-plane, as in Fig. 4.

The symbols $[m, n]$ in Fig. 4 have the following meanings. Consider, for example, (a_x, a_y, a_z) lies in regions (VIII), (i) and (1) as in Fig. 3, Eqs. (85) and (86). This situation is expressed as (VIII)-(i)-(1), and considered $a_x < a_y < a_z < 0$

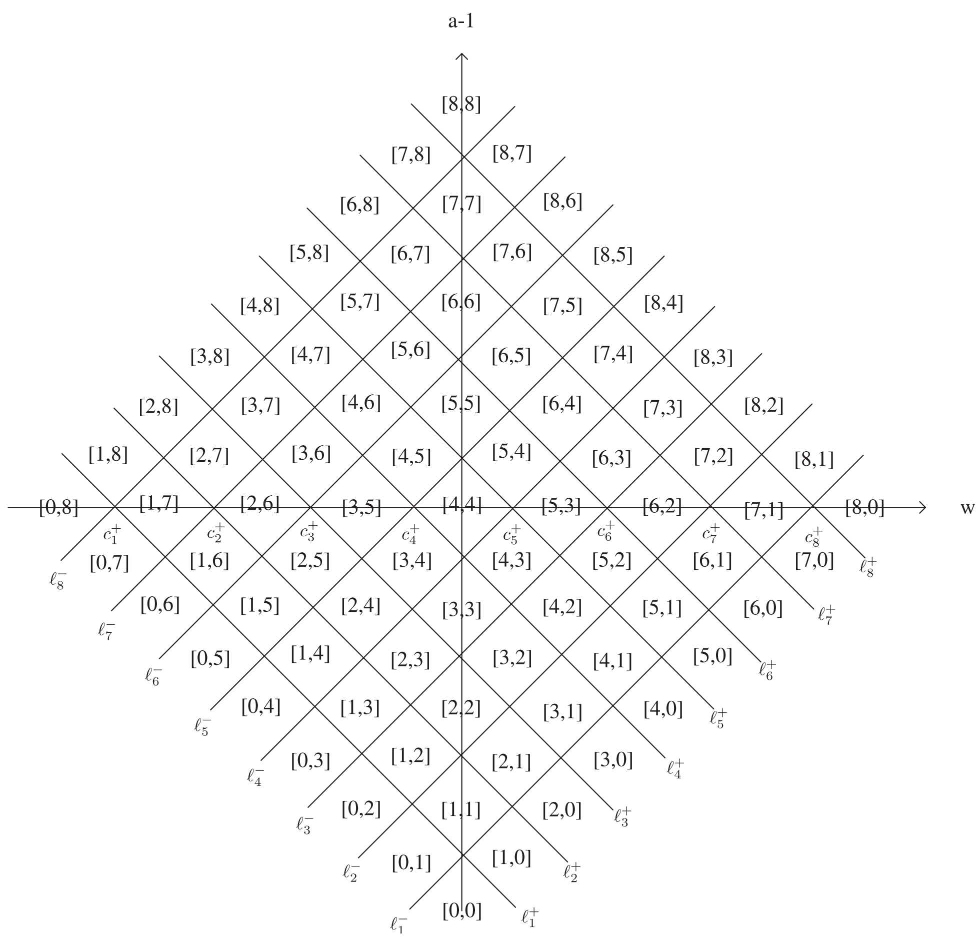


Fig. 4. Partition of $(w, a-1)$-plane.

and $|a_x| > |a_y| + |a_z|$. Denoted by

Table 1. The intersects of ℓ_i^+ and ℓ_j^-.

	$(v_{i+1,j,k}, v_{i,j+1,k}, v_{i,j,k+1})$	$-(a_x v_{i+1,j,k} + a_y v_{i,j+1,k} + a_z v_{i,j,k+1})$
$c_1^+ = c_8^-$	$(-1,-1,-1)$	$a_x + a_y + a_z$
$c_2^+ = c_7^-$	$(-1,-1,1)$	$a_x + a_y - a_z$
$c_3^+ = c_6^-$	$(-1,1,-1)$	$a_x - a_y + a_z$
$c_4^+ = c_5^-$	$(-1,1,1)$	$a_x - a_y - a_z$
$c_5^+ = c_4^-$	$(1,-1,-1)$	$-a_x + a_y + a_z$
$c_6^+ = c_3^-$	$(1,-1,1)$	$-a_x + a_y - a_z$
$c_7^+ = c_2^-$	$(1,1,-1)$	$-a_x - a_y + a_z$
$c_8^+ = c_1^-$	$(1,1,1)$	$-a_x - a_y - a_z$

Then, $c_8^+ > c_7^+ > c_6^+ > c_5^+ > 0 > c_4^+ > c_3^+ > c_2^+ > c_1^+ >$ are the intersects of ℓ_i^+ and ℓ_j^- on the w-axis displayed in Fig. 4.

With reference to the local patterns on cube-cells, +1 is represented by the symbol + and −1 is represented by the symbol −. The 2^4 local patterns can be listed and ordered, as in Fig. 5.

Now, when $(w, a-1)$ lies in region $[m, n]$ in Fig. 4, the only admissible patterns are exactly ①, ②, …, ⓜ and ①′, ②′, …, ⓝ′. For instance, in region (VIII)-(i)-(1) and $(a-1, w) \in [4, 8]$ only ①, ②, ③, ④ and ①′, ②′, ③′, ④′⑤′, ⑥′⑦′, ⑧′ can be produced. This fact is equivalent to the holding of inequalities in Eqs. (83) and (84) if and only if $v_{i,j,k}$, $v_{i+1,j,k}$, $v_{i,j+1,k}$ and $v_{i,j,k+1}$ are of the form ①, ②, ③, ④ and ①′, ②′, ③′, ④′⑤′, ⑥′⑦′, ⑧′.

Next, the transition matrix of local patterns in region (VIII)-(i)-(1)-[4,8] can be derived as

$$\mathbb{A}_{x;2\times2\times2} = G \otimes E \otimes E \otimes E.$$

Then, according to Proposition 3.9, the admissible local patterns in $\Sigma_{2\times m_2\times m_3}$ and its corresponding transition matrices are

$$\mathbb{A}_{x;2\times m_2\times 2} = \otimes(G \otimes E)^{m_2-1} \otimes (\otimes E^2),$$

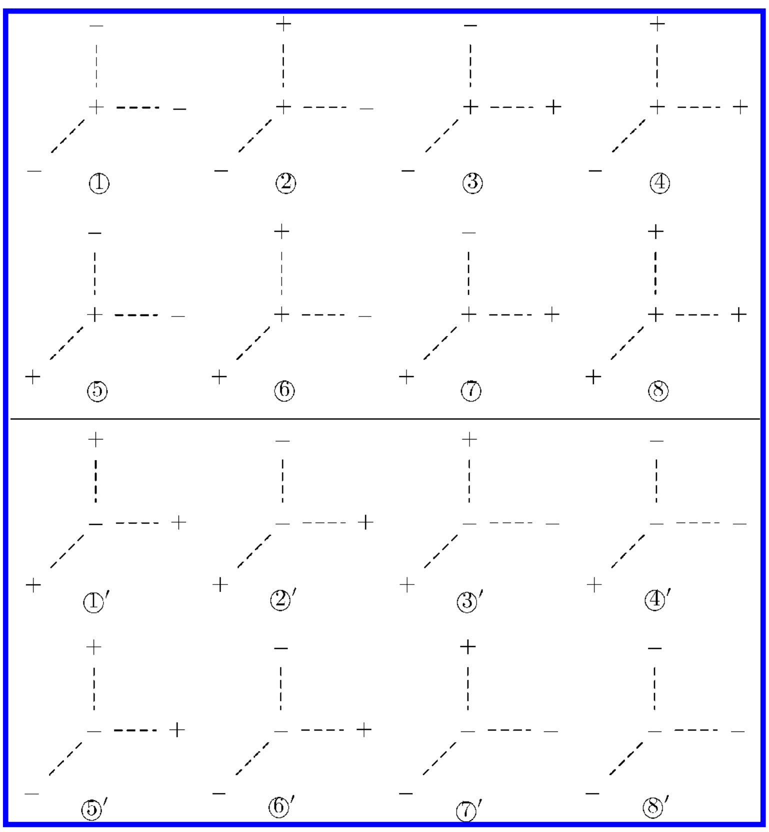


Fig. 5. Ordering of local patterns in partition (VIII)-(i)-(1).

$$\mathbb{A}_{\hat{x};2\times m_2\times 2} = (\otimes G^{m_2-1}) \otimes (\otimes E^{m_2+1}),$$
$$\mathbb{A}_{\hat{x};2\times m_2\times m_3} = \otimes((\otimes G^{m_2-1}) \otimes E)^{m_3-1} \otimes (\otimes E^{m_2}),$$

as in Eqs. (43), (44) and (45).

Finally, the connecting operator is adopted to examine the complexity of the set of mosaic patterns in 3DCNN. That is, the lower bound of spatial entropy in the region (VIII)-(i)-(1)-[4,8] can be estimated.

Proposition 5.1. *Consider* $\mathbb{A}_{x;2\times 2\times 2} = G \otimes E \otimes E \otimes E$, *then*

$$S_{z;m_1;m_2 2;11} = C_{z;m_1;m_2 2;11} = (\otimes G^{m_2-1}) \otimes E,$$
$$\rho(S_{z;m_1;m_2 2;11}) = 2g^{m_2-1}$$

and

$$h(\mathbb{A}_{x;2\times 2\times 2}) \geq \frac{1}{2}\log g,$$

where $g = (1+\sqrt{5})/2$ *is the golden-mean. Moreover, since*

$$\mathbb{A}_{\hat{x};2\times m_2\times m_3} = \otimes((\otimes G^{m_2-1}) \otimes E)^{m_3-1} \otimes (\otimes E^{m_2})$$

and

$$\rho(\mathbb{A}_{\hat{x};2\times m_2\times m_3}) = 2^{m_2+m_3-1} g^{(m_2-1)(m_3-1)},$$

the spatial entropy can be exactly computed as

$$h(\mathbb{A}_{x;2\times 2\times 2}) = \log g$$

as in Proposition 3.9.

Proof. According to Eq. (44),

$$\mathbb{A}_{\hat{x};2\times m_2\times 2} = (\otimes G^{m_2-1}) \otimes (\otimes E^{m_2+1})$$

is obtained. Evidently,

$$A_{\hat{x};2\times m_2\times 2;1} = \otimes E^{m_2}$$

and

$$(\mathbb{A}^{(r)}_{\hat{x};2\times m_2\times 2})^{(c)}_{;1} = (\otimes G^{m_2-1}) \otimes E.$$

By Remark 4.3, the connecting operator

$$C_{z;m_1;m_2 2;11} = A_{\hat{x};2\times m_2\times 2;1} \circ (\mathbb{A}^{(r)}_{\hat{x};2\times m_2\times 2})^{(c)}_{;1} = (\otimes G^{m_2-1}) \otimes E.$$

Therefore, based on Remark 4.13, the lower bound of spatial entropy is estimated as

$$h(\mathbb{A}_{x;2\times 2\times 2}) \geq \lim_{m_2\to\infty} \frac{1}{2m_2}\log \rho(S_{z;m_1;m_2 2;11}) = \lim_{m_2\to\infty} \frac{\log 2g^{m_2-1}}{2m_2} = \frac{1}{2}\log g.$$

■

Remark 5.2. For the general template $A = (a_{\alpha,\beta,\gamma})$ where $a_{\alpha,\beta,\gamma} \neq 0$, the basic set in $\Sigma_{3\times 3\times 3}$ must be extended to the basic set in $\Sigma_{4\times 4\times 4}$. Then, the method described above can be applied, as stated in Remark 4.14. The details are omitted here for brevity.

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