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張量積的數值域

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英文關鍵詞：

矩陣張量乘積的數值半徑
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中文摘要：

　　假設 A 和 B 分別是 n 階和 m 階的矩陣，則 $w(A \otimes B) \leq \|A\| w(B)$ 早已知成立，

其中 $w(\cdot)$ 和 $\|\cdot\|$ 分別表示相關矩陣的數值半徑和算子範數。在本篇論文中，我們考

慮此不等式何時成爲一等式。我們證明：(一)假設 $\|A\| = 1$ 且 $w(A \otimes B) = w(B)$，則

下列二條件之一成立：(i)A 有一酉部份，(ii)A 完全沒有酉部份且 B 的數值域 $w(B)$

是一個中心點爲原點的圓盤，(二)設 $\|A\| = \|A^k\| = 1$，其中 k 是某一固定的正整數，

則 $w(A) \geq \cos(\pi/(k+2))$。並且此處等號成立的充份必要條件是 A 酉相似於 J_{k+1} 和

B 的直和，這裡 J_{k+1} 表示 $k+1$ 階的約當塊，而 B 係一矩陣滿足

$w(B) \leq \cos(\pi/(k+2))$，且(三)假設 B 是一非負矩陣且其實部份是排列不可約的，

則 $w(A \otimes B) = \|A\| w(B)$ 成立的充份必要條件是 $p_A = \infty$ 或者 $n_B \leq p_A < \infty$ 且 B 排列

相似於一個塊狀移位矩陣，其中 p_A 係使得 $\|A^l\| = \|A\|^l$ 成立的最大 l，而 n_B 是使得

B^l 不爲零的最大 l。

關鍵字：數值域、數值半徑、張量積、S_n-矩陣、非負矩陣。

Numerical Radii for Tensor Products of Matrices

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For n-by-n and m-by-m complex matrices A and B, it is known that the inequality $w(A \otimes B) \leq \|A\| w(B)$ holds, where $w(\cdot)$ and $\|\cdot\|$ denote, respectively, the numerical radius and the operator norm of a matrix. In this paper, we consider when this becomes an equality. We show that (1) if $\|A\| = 1$ and $w(A \otimes B) = w(B)$, then one of the following two conditions holds: (i) A has a unitary part, and (ii) A is completely nonunitary and the numerical range $W(B)$ of B is a circular disc centered at the origin, (2) if $\|A\| = \|A^k\| = 1$ for some k, $1 \leq k < \infty$, then $w(A) \geq \cos(\pi/(k + 2))$, and, moreover, the equality holds if and only if A is unitarily similar to the direct sum of the $(k+1)$-by-$(k+1)$ Jordan block J_{k+1} and a matrix B with $w(B) \leq \cos(\pi/(k + 2))$, and (3) if B is a nonnegative matrix with its real part (permutationally) irreducible, then $w(A \otimes B) = \|A\| w(B)$ if and only if either $p_A = \infty$ or $n_B \leq p_A < \infty$ and B is permutationally similar to a block-shift matrix

$$
\begin{bmatrix}
0 & B_1 & & \\
 & 0 & \ddots & \\
 & & \ddots & B_k \\
 & & & 0
\end{bmatrix}
$$

with $k = n_B$, where $p_A = \sup\{\ell \geq 1 : \|A^\ell\| = \|A\|^\ell\}$ and $n_B = \sup\{\ell \geq 1 : B^\ell \neq 0\}$.

Keywords: numerical range; numerical radius; tensor product; S_n-matrix; nonnegative matrix

AMS Subject Classifications: 15A60; 15A69; 15B48

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1.   Introduction and Preliminaries

For any n-by-n complex matrix A, its *numerical range* $W(A)$ is, by definition, the subset $\{\langle Ax, x \rangle : x \in \mathbb{C}^n, \|x\| = 1\}$ of the complex plane $\mathbb{C}$, where $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ denote the standard inner product and its associated norm in $\mathbb{C}^n$, respectively. The *numerical radius* $w(A)$ of A is $\max\{|z| : z \in W(A)\}$. It is known that $W(A)$ is a nonempty compact convex subset of $\mathbb{C}$, and $w(A)$ satisfies $\|A\|/2 \leq w(A) \leq \|A\|$, where $\|A\|$ denotes the usual operator norm of A. For other properties of the numerical range and numerical radius, the reader may consult [7], [9, Chapter 22] or [12, Chapter 1].

The *tensor product* (or *Kronecker product*) $A \otimes B$ of an n-by-n matrix $A = [a_{ij}]_{i,j=1}^n$ and an m-by-m matrix B is the (mn)-by-(mn) matrix

$$\begin{bmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & & \vdots \\ a_{n1}B & \cdots & a_{nn}B \end{bmatrix}.$$

It is known that $A \otimes B$ and $B \otimes A$ are unitarily similar and $\|A \otimes B\| = \|A\| \cdot \|B\|$. Other properties of the tensor product can be found in [12, Chapter 4].

The main concern of this paper is the relations between the numerical radius of $A \otimes B$ and those of A and B. For one direction, we have $w(A \otimes B) \leq \min\{\|A\|w(B), \|B\|w(A)\}$. This can be proven by using the unitary dilation of contractions, as to be done below. On the other hand, we also have $w(A \otimes B) \geq w(A)w(B)$. We are interested in when these become equalities. In the present paper, we obtain various conditions, necessary or sufficient, for $w(A \otimes B) = \|A\|w(B)$ to hold. The discussions on the equality $w(A \otimes B) = w(A)w(B)$ will be the subject of a subsequent paper of ours.

For the ease of exposition, we introduce two indices for an n-by-n matrix A: the *power norm index* p_A and *nilpotency index* n_A of A. They are defined, respectively, by

$$p_A = \sup\{k \geq 1 : \|A^k\| = \|A\|^k\}$$

and

$$n_A = \begin{cases} \sup\{k \geq 1 : A^k \neq 0_n\} & \text{if } A \neq 0_n, \\ 0 & \text{if } A = 0_n, \end{cases}$$

where 0_n denotes the n-by-n zero matrix.

We start in Section 2 by proving that if $\|A\| = 1$ and $w(A \otimes B) = w(B)$, then either (i) A has a unitary part, or (ii) A is completely nonunitary and $W(B)$ is a circular disc centered at the origin (Theorem 2.2). The proof depends on the dilation of A to a direct sum of S_ℓ-matrices with $\ell \leq n$, the Poncelet property of the numerical ranges of matrices of the latter class, and Anderson's theorem on the circular disc numerical range. As a by-product, we obtain a lower bound for $w(A)$ when A satisfies $\|A\| = \|A^k\| = 1$ for some k, $1 \leq k < n$: $w(A) \geq \cos(\pi/(k+2))$, and determine exactly when this bound is attained: this is the case if and only if A is unitarily similar to $J_{k+1} \oplus B$, where J_{k+1} is the $(k+1)$-by-$(k+1)$ *Jordan block*

$$\begin{bmatrix} 0 & 1 & & \\ & 0 & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{bmatrix}$$

and B is a finite matrix with $w(B) \leq \cos(\pi/(k+2))$ (Theorem 2.10). This generalizes the classical result of Willams and Crimmins [17] for $k = 1$. We conclude Section 2 with a result on nilpotent contractions, namely, we prove that if A is an n-by-n matrix with $\|A\| = 1$, then a necessary and sufficient condition for $p_A = n_A < \infty$ to hold is that A be unitarily similar to a direct sum $J_{k+1} \oplus B$, where $k = p_A$ and $B^{k+1} = 0$ (Theorem 2.13).

Finally, in Section 3, we consider B to be a nonnegative matrix with $\mathrm{Re}\, B$ ($= (B + B^*)/2$) (permutationally) irreducible. We obtain in Theorem 3.1 a complete characterization for $w(A \otimes B) = \|A\|w(B)$, namely, this is the case if and only if either $p_A = \infty$ or $n_B \leq p_A < \infty$ and B is permutationally similar to a block-shift matrix of the form

$$\begin{bmatrix} 0 & B_1 & & \\ & 0 & \ddots & \\ & & \ddots & B_k \\ & & & 0 \end{bmatrix}$$

with $k = n_B$.

As was mentioned before, the inequality $w(A \otimes B) \leq \|A\|w(B)$ for n-by-n and m-by-m matrices A and B is known. It is a consequence of [10, Theorem 3.4] because $A \otimes B$ is the product of $A \otimes I_m$ and $I_n \otimes B$, and the latter two matrices *doubly*

commute, that is, $A \otimes I_m$ commutes with both $I_n \otimes B$ and its adjoint $I_n \otimes B^*$. Here we give a simple proof based on the unitary dilation of contractions.

Proposition 1.1. *If A and B are n-by-n and m-by-m matrices, respectively, then*
$$w(A \otimes B) \leq \min\{\|A\|w(B), \|B\|w(A)\}.$$

Proof. We need only prove that $w(A \otimes B) \leq \|A\|w(B)$, and may assume that $\|A\| = 1$. Then the $(2n)$-by-$(2n)$ matrix

$$U = \begin{bmatrix} A & (I_n - AA^*)^{1/2} \\ (I_n - A^*A)^{1/2} & -A^* \end{bmatrix}$$

is unitary. Let U be unitarily similar to the diagonal matrix $\mathrm{diag}\,(u_1, \ldots, u_{2n})$, where $|u_j| = 1$ for all j. Then

$$w(A \otimes B) \leq w(U \otimes B) = w(\sum_{j=1}^{2n} \oplus u_j B) = \max_j w(u_j B) = w(B) = \|A\|w(B). \quad \square$$

We conclude this section with some basic properties of the indices p_A and n_A of a matrix A.

Proposition 1.2. *Let A be an n-by-n matrix. Then*

  (a) $1 \leq p_A \leq n - 1$ *or* $p_A = \infty$,

  (b) $p_A = n - 1$ *if and only if A is a nonzero multiple of a S_n-matrix, and*

  (c) *the following conditions are equivalent:*

       (1) $p_A = \infty$,

       (2) $\|A\| = \rho(A)$,

       (3) $\|A\| = w(A)$,

       and if $\|A\| = 1$, then the above are also equivalent to

       (4) *A has a unitary part.*

Here $\rho(A)$ denotes the *spectral radius* of A, that is, $\rho(A)$ is the maximum modulus of the eigenvalues of A.

Recall that an n-by-n matrix A is of *class S_n* if it is a contraction ($\|A\| \leq 1$), its eigenvalues are all in $\mathbb{D} \equiv \{z \in \mathbb{C} : |z| < 1\}$, and $\mathrm{rank}\,(I_n - A^*A) = 1$. Any contraction A is unitarily similar to the direct sum of a unitary matrix U, called the *unitary part* of A, and a completely nonunitary contraction A', called the *c.n.u. part* of A. The latter means that A' is not unitarily similar to any direct sum with a unitary summand.

4

Proof of Proposition 1.2. (a) was obtained by Pták in 1960 (cf [15, Theorem 2.1]) and (b) was proven in [4, Theorem 3.1]. As for (c), the implication $(1) \Rightarrow (2)$ is by [9, Problem 88], $(2) \Rightarrow (3)$ by the known inequalities $\rho(A) \leq w(A) \leq \|A\|$, $(3) \Rightarrow (2)$ by [9, Problem 218 (b)], and $(2) \Rightarrow (1)$ by the inequalities $\rho(A) \leq \|A^k\|^{1/k} \leq \|A\|$ for all $k \geq 1$. If $\|A\| = \rho(A) = 1$, then, letting λ be an eigenvalue of A with $|\lambda| = 1$, we have the unitary similarity of A and a matrix of the form $\begin{bmatrix} \lambda & B \\ 0 & C \end{bmatrix}$. Since $\|A\| = |\lambda| = 1$ implies that $B = 0$, A is unitarily similar to $[\lambda] \oplus C$ and thus has a unitary part. This proves $(2) \Rightarrow (4)$. That $(4) \Rightarrow (2)$ is trivial. $\qquad\square$

Proposition 1.3. *Let A be an n-by-n matrix. Then*

(a) $0 \leq n_A \leq n - 1$ *or* $n_A = \infty$,

(b) $n_A = n - 1$ *if and only if A is similar to the n-by-n Jordan block J_n,*

(c) $n_A = \infty$ *if and only if A is not nilpotent, and*

(d) $p_A \leq n_A$ *for $A \neq 0_n$.*

We omit its easy proofs.

In the following, we use $\sigma(A)$ to denote the *spectrum* of A, that is, $\sigma(A)$ is the set of eigenvalues of A. An n-by-n matrix A is a *dilation* of an m-by-m matrix B (or B is a *compression* of A) if there is an n-by-m matrix V such that $B = V^*AV$ and $V^*V = I_m$. This is equivalent to A being unitarily similar to a matrix of the form $\begin{bmatrix} B & * \\ * & * \end{bmatrix}$.

2.   Contractions

We start with a simple condition which yields the equality $w(A \otimes B) = \|A\|w(B)$.

Lemma 2.1. *If A is an n-by-n matrix with $p_A = \infty$, then $w(A \otimes B) = \|A\|w(B)$ for any m-by-m matrix B. In particular, this is the case for A a contraction with a unitary part.*

Proof. Since $p_A = \infty$ implies, by Proposition 1.2 (c), that $\|A\| = w(A)$. If λ is a number in $W(A)$ with $|\lambda| = w(A)$, then $|\lambda| = \|A\|$. Since A is unitarily similar to a matrix of the form $\begin{bmatrix} \lambda & * \\ * & * \end{bmatrix}$, we have the unitary similarity of $A \otimes B$ and $\begin{bmatrix} \lambda B & * \\ * & * \end{bmatrix}$. It follows that $\|A\|w(B) = w(\lambda B) \leq w(A \otimes B)$. On the other hand, we also have $w(A \otimes B) \leq \|A\|w(B)$ by Proposition 1.1. Thus $w(A \otimes B) = \|A\|w(B)$ holds. $\quad\square$

The next theorem is one of the main results of this section. It gives a necessary condition for the equality $w(A \otimes B) = \|A\| w(B)$.

Theorem 2.2. *Let A and B be n-by-n and m-by-m matrices, respectively. If $\|A\| = 1$ and $w(A \otimes B) = w(B)$, then one of the following two conditions holds:* (i) *A has a unitary part, and* (ii) *A is c.n.u. and $W(B)$ is a circular disc centered at the origin.*

We first prove this for the case when A is an S_n-matrix. The numerical ranges of such matrices are known to have the *Poncelet property*, namely, if A is of class S_n, then, for any point λ on the unit circle $\partial \mathbb{D}$, there is a unique (up to unitary similarity) $(n+1)$-by-$(n+1)$ unitary dilation U of A such that λ is an eigenvalue of U and each edge of the $(n+1)$-gon $\partial W(U)$ intersects $W(A)$ at exactly one point (cf. [2, Theorem 2.1 and Lemma 2.2]).

Lemma 2.3. *Let A be an S_n-matrix and B an m-by-m matrix. If $w(A \otimes B) = w(B)$, then $W(B)$ is a circular disc centered at the origin.*

Proof. Let $U_1, \ldots, U_{m+1}$ be $(n+1)$-by-$(n+1)$ unitary dilations of A with $\sigma(U_i) \cap \sigma(U_j) = \emptyset$ for all i and j, $1 \leq i \neq j \leq m+1$. We may assume that $U_j = \mathrm{diag}\,(\lambda_{1j}, \ldots, \lambda_{n+1,j})$ for each j, where $|\lambda_{ij}| = 1$ for all i and j. Let V_j be an $(n+1)$-by-n matrix such that $A = V_j^* U_j V_j$ and $V_j^* V_j = I_n$ for each j. Since $\|A\| = 1$ and

$$w(A \otimes \lambda B) = w(A \otimes B) = w(B) = w(\lambda B)$$

for any λ, $|\lambda| = 1$, we may further assume that $w(B)$ is in $W(A \otimes B)$. Let x be a unit vector in $\mathbb{C}^n \otimes \mathbb{C}^m$ such that $\langle (A \otimes B)x, x \rangle = w(B)$. We decompose $(V_j \otimes I_m)x$ as $y_{1j} \oplus \cdots \oplus y_{n+1,j}$ with y_{ij}, $1 \leq i \leq n+1$, in $\mathbb{C}^m$ for each j. Then

$$
\begin{aligned}
w(B) &= \langle (A \otimes B)x, x \rangle \\
&= \langle (U_j \otimes B)(V_j \otimes I_m)x, (V_j \otimes I_m)x \rangle \\
&= \langle (\lambda_{1j}B \oplus \cdots \oplus \lambda_{n+1,j}B)(y_{1j} \oplus \cdots \oplus y_{n+1,j}), y_{1j} \oplus \cdots \oplus y_{n+1,j} \rangle \\
&= \sum_{i=1}^{n+1} \langle \lambda_{ij} B y_{ij}, y_{ij} \rangle \\
&\leq \sum_{i=1}^{n+1} |\langle B y_{ij}, y_{ij} \rangle|.
\end{aligned}
$$

6

Letting $\eta_{ij} = \langle B(y_{ij}/\|y_{ij}\|), y_{ij}/\|y_{ij}\| \rangle$ for each $y_{ij} \neq 0$, we obtain

$$w(B) = \sum_{y_{ij} \neq 0} \lambda_{ij} \|y_{ij}\|^2 \eta_{ij} \leq \sum_{y_{ij} \neq 0} \|y_{ij}\|^2 |\eta_{ij}| \leq \sum_{y_{ij} \neq 0} \|y_{ij}\|^2 w(B) = w(B)$$

since

$$\sum_i \|y_{ij}\|^2 = \|(V_j \otimes I_m)x\|^2 = \|x\|^2 = 1.$$

Thus we have equalities throughout the above sequence, which yields that $w(B) = \lambda_{ij}\eta_{ij}$ for $y_{ij} \neq 0$. Since $\sum_i \|y_{ij}\|^2 = 1$, this must hold for at least one i, say, i_j. Hence $\overline{\lambda}_{i_j j} w(B) = \eta_{i_j j}$ is in $\partial W(B)$ for each j. Note that such $\overline{\lambda}_{i_j j} w(B)$'s, $1 \leq j \leq m+1$, are distinct from each other by our assumption on the disjointness of the spectra of the U_j's. This shows that the boundary of $W(B)$ and the circle $|z| = w(B)$ intersect at at least $m+1$ points. Since $W(B)$ is contained in $\{z \in \mathbb{C} : |z| \leq w(B)\}$, we apply Anderson's theorem (cf. [3, Theorem] or [20]) to infer that $W(B) = \{z \in \mathbb{C} : |z| \leq w(B)\}$. □

Proof of Theorem 2.2. We assume that A is c.n.u. Then A can be dilated to the direct sum $A' \oplus \cdots \oplus A'$ of rank $(I_n - A^*A)$ many copies of some S_ℓ-matrix A' with $\ell \leq n$ (cf. [18, Theorem 1.4] or [21, Lemma 3 (a)]). Hence $A \otimes B$ dilates to $(A' \oplus \cdots \oplus A') \otimes B = (A' \otimes B) \oplus \cdots \oplus (A' \otimes B)$. We have

$$w(B) = w(A \otimes B) \leq w((A' \otimes B) \oplus \cdots \oplus (A' \otimes B)) = w(A' \otimes B) \leq \|A'\| w(B) = w(B).$$

Thus $w(A' \otimes B) = w(B)$. It follows from Lemma 2.3 that $W(B)$ is a circular disc centered at the origin. □

An easy consequence of Theorem 2.2 is that the converse of Lemma 2.1 is also true.

Corollary 2.4. *For an n-by-n matrix A, the equality $w(A \otimes B) = \|A\| w(B)$ holds for all matrices B if and only if $p_A = \infty$.*

Proof. For the necessity, assume that $\|A\| = 1$ and let B be any matrix with its numerical range not a circular disc centered at the origin. Theorem 2.2 yields that A has a unitary part. Then $p_A = \infty$ follows immediately. □

In Theorem 2.2, if B is the Jordan block J_m, then we have the following characterizations for $w(A \otimes B) = \|A\| w(B)$.

Theorem 2.5. *Let A be an n-by-n matrix with $\|A\| = 1$. Then the following conditions are equivalent:*

(a) $W(A \otimes J_m) = W(J_m)$,

(b) $w(A \otimes J_m) = w(J_m)$,

(c) $A \otimes J_m$ *is unitarily similar to $J_m \oplus B$ for some matrix B with $w(B) \leq w(J_m)$, and*

(d) $\|A^{m-1}\| = 1$.

If, in addition, $n = m$, then the above conditions are also equivalent to

(e) *either A has a unitary part or A is of class S_n, and*

(f) $p_A = \infty$ *or $n - 1$.*

Note that $W(J_m) = \{z \in \mathbb{C} : |z| \leq \cos(\pi/(m+1))\}$ (cf. [8, Proposition 1]).

Proof of Theorem 2.5. The implication (a) $\Rightarrow$ (b) is trivial. To prove (b) $\Rightarrow$ (c), note that $(A \otimes J_m)^m = A^m \otimes J_m^m = 0_{nm}$ and $\|A \otimes J_m\| = \|A\|\|J_m\| = 1$. If x is a unit vector in $\mathbb{C}^n \otimes \mathbb{C}^m$ such that $|\langle (A \otimes J_m)x, x \rangle| = w(A \otimes J_m)$, then $w(A \otimes J_m) = w(J_m) = \cos(\pi/(m+1))$ implies that the subspace K of $\mathbb{C}^n \otimes \mathbb{C}^m$ generated by the vectors $x, (A \otimes J_m)x, \ldots, (A \otimes J_m)^{m-1}x$ is reducing for $A \otimes J_m$, and the restriction of $A \otimes J_m$ to K is unitarily similar to J_m (cf. [8, Theorem 1 (2)]). Hence $A \otimes J_m$ is unitarily similar to $J_m \oplus B$, where B is the restriction of $A \otimes J_m$ to $K^\perp$. We obviously have $w(B) \leq w(A \otimes J_m) = w(J_m)$.

For (c) $\Rightarrow$ (d), note that $A^{m-1} \otimes J_m^{m-1}$ is unitarily similar to $J_m^{m-1} \oplus B^{m-1}$ under (c). Hence

$$\|A^{m-1}\| = \|A^{m-1} \otimes J_m^{m-1}\| = \|J_m^{m-1} \oplus B^{m-1}\| = \max\{\|J_m^{m-1}\|, \|B^{m-1}\|\} = 1.$$

To prove (d) $\Rightarrow$ (c), let x be a unit vector in $\mathbb{C}^n$ such that $\|A^{m-1}x\| = 1$. Then $\|A^{m-j}x\| = 1$ for all j, $1 \leq j \leq m$. Let $\{e_1, \ldots, e_m\}$ be the standard basis for $\mathbb{C}^m$, let $x_j = A^{m-j}x \otimes e_j$, $1 \leq j \leq m$, and let K be the subspace of $\mathbb{C}^n \otimes \mathbb{C}^m$ generated by $x_1, \ldots, x_m$. Then $(A \otimes J_m)x_1 = 0$ and $(A \otimes J_m)x_j = x_{j-1}$ for $2 \leq j \leq m$. Since $\{x_1, \ldots, x_m\}$ is an orthonormal basis of K, this shows that $(A \otimes J_m)K \subseteq K$ and the restriction of $A \otimes J_m$ to K is unitarily similar to J_m. On the other hand, it follows from $\|A \otimes J_m\| = \|A\|\|J_m\| = 1$ and

$$(A \otimes J_m)^* x_m = (A^* \otimes J_m^*)(x \otimes e_m) = (A^*x) \otimes (J_m^* e_m) = (A^*x) \otimes 0 = 0$$

that K is reducing for $A \otimes J_m$, and hence $A \otimes J_m$ is unitarily similar to $J_m \oplus B$,

where B is the restriction of $A \otimes J_m$ to $K^{\perp}$. Obviously, we have

$$w(B) \leq w(A \otimes J_m) \leq \|A\| w(J_m) = w(J_m).$$

To prove (c) $\Rightarrow$ (a), note that the unitary similarity of J_m and $e^{i\theta} J_m$ for all real θ implies the same for $A \otimes J_m$ and $e^{i\theta}(A \otimes J_m)$. Thus $W(A \otimes J_m)$ is a circular disc centered at the origin. (c) implies that $w(A \otimes J_m) = w(J_m)$, which means that the radii of the two circular discs $W(A \otimes J_m)$ and $W(J_m)$ are equal. Therefore, $W(A \otimes J_m) = W(J_m)$ holds.

Now assume that $n = m$ and that $\|A^{n-1}\| = 1$. If $\|A^n\| = 1$, then $p_A = \infty$ and hence A has a unitary part by Proposition 1.2 (a) and (c). On the other hand, if $\|A^n\| < 1$, then A is of class S_n by [4, Theorem 3.1]. This shows that (d) $\Rightarrow$ (e). Next, if (e) is true, then $p_A = \infty$ or $n - 1$ depending on whether A has a unitary part or A is of class S_n (cf. [4, Theorem 3.1] for the latter). This proves (f). Finally, if $p_A = \infty$, then $\|A^k\| = 1$ for all $k \geq 1$, and, in particular, $\|A^{n-1}\| = 1$. On the other hand, if $p_A = n - 1$, then $\|A^{n-1}\| = \|A\|^{n-1} = 1$. This proves (f) $\Rightarrow$ (d). $\square$

The next proposition gives a characterization of $w(A \otimes B) = \|A\| w(B)$ when B is of class S_m.

Proposition 2.6. *Let A be an n-by-n matrix with $\|A\| = 1$, and B be an S_m-matrix. Then $w(A \otimes B) = w(B)$ if and only if either* (i) *A has a unitary part, or* (ii) *A is c.n.u., $\|A^{m-1}\| = 1$ and B is unitarily similar to J_m.*

Its proof depends on a special property of S_n-matrices. The following lemma is from [19, Lemma 5]. Here we give a shorter geometric proof.

Lemma 2.7. *Let A be an S_n-matrix. Then $W(A)$ is a circular disc centered at the origin if and only if A is unitarily similar to J_n.*

Proof. If $W(A)$ is as asserted, then the Poncelet property of $W(A)$ says that it is circumscribed by $(n + 1)$-gons with vertices on the unit circle. As the circular disc $\{z \in \mathbb{C} : |z| \leq \cos(\pi/(n + 1))\}(= W(J_n))$ is circumscribed by any regular $(n + 1)$-gon on the unit circle, if the radius of $W(A)$ is not equal to $\cos(\pi/(n + 1))$, then we infer from a geometrical consideration that $W(A)$ cannot have the Poncelet property. Thus $W(A)$ must equal $W(J_n)$. The unitary similarity of A and J_n then follows from [2, Theorem 3.2]. The converse is trivial. $\square$

Proof of Proposition 2.6. If $w(A \otimes B) = w(B)$, then, by Theorem 2.2, either A has a unitary part or A is c.n.u. and $W(B)$ is a circular disc centered at the origin. In the latter case, Lemma 2.7 yields the unitary similarity of B and J_m, and then Theorem 2.5 gives $\|A^{m-1}\| = 1$. The converse also follows from Theorem 2.5. $\quad\square$

Note that, under the conditions of Proposition 2.6, if A is c.n.u., then we automatically have $m \leq n$. This is because if, otherwise, $m > n$, then $\|A^{m-1}\| = 1$ yields, by Proposition 1.2 (a) and (c), that A has a unitary part.

A specific example of the results obtained so far is in the next proposition.

Proposition 2.8. *Let n and m be positive integers. Then $W(J_n \otimes J_m) = W(J_\ell)$, where $\ell = \min\{n, m\}$, and thus $w(J_n \otimes J_m) = \min\{w(J_n), w(J_m)\}$.*

Proof. Assume that $m \leq n$. Since the principal submatrix of $J_n \otimes J_m$ formed by its rows and columns numbered $1, m+2, 2m+3, \ldots$, and $(m-1)m + m$ is J_m, we have that $J_n \otimes J_m$ is a dilation of J_m. Thus $w(J_m) \leq w(J_n \otimes J_m)$. The reversed inequality $w(J_n \otimes J_m) \leq \|J_n\| w(J_m) = w(J_m)$ is by Proposition 1.1. Therefore, $w(J_n \otimes J_m) = w(J_m)$ holds. As was seen in the proof of (c) $\Rightarrow$ (a) in Theorem 2.5, $W(J_n \otimes J_m)$ is a circular disc centered at the origin. Thus the equality of $w(J_n \otimes J_m)$ and $w(J_m)$ implies that of $W(J_n \otimes J_m)$ and $W(J_m)$. $\quad\square$

Besides S_n-matrices, another generalization of the Jordan blocks is the companion matrices. Recall that a companion matrix is one of the form

$$\begin{bmatrix} 0 & 1 & & & & \\ & 0 & 1 & & & \\ & & \cdot & \cdot & & \\ & & & \cdot & \cdot & \\ & & & & \cdot & \cdot \\ & & & & 0 & 1 \\ -a_n & -a_{n-1} & \cdots & -a_2 & -a_1 \end{bmatrix},$$

whose characteristic and minimal polynomials are both equal to $z^n + \sum_{j=1}^n a_j z^{n-j}$. The numerical ranges of such matrices have been studied in [1, 5, 6].

Proposition 2.9. *Let A be an n-by-n ($n \geq 2$) companion matrix. Then the following conditions are equivalent:*

(a) $w(A \otimes A) = \|A\| w(A)$,

(b) *one of the following holds:* (i) *A is unitary,* (ii) *$A = J_n$, or* (iii) *A is unitarily similar to a direct sum $[a\omega_n^j] \oplus B$, where $|a| > 1$, $\omega_n = e^{i(2\pi/n)}$, $0 \le j \le n-1$, and B is an S_{n-1}-matrix with eigenvalues $(1/\bar{a})\omega_n^k$, $0 \le k \le n-1$ and $k \ne j$, and*

(c) $p_A = n_A = \infty$ *or* $n - 1$.

Proof. To prove (a) $\Rightarrow$ (b), let $A' = A/\|A\|$. Then (a) gives $w(A' \otimes A') = w(A')$. By Theorem 2.2, either A' has a unitary part or it is c.n.u. with numerical range a circular disc centered at the origin. In the former case, either A is normal or is unitarily similar to a matrix of the form $[a\omega_n^j] \oplus B$, where $|a| = \|A\| \ge 1$ and B is of size $n - 1$ with eigenvalues $(1/\bar{a})\omega_n^k$, $0 \le k \le n - 1$ and $k \ne j$ (cf. [5, Theorem 1.1 and Corollary 1.3]). If A is normal or $|a| = 1$, then A is unitary by [5, Corollary 1.2]. Hence we may assume that $|a| > 1$. Thus the eigenvalues of B are all contained in $\mathbb{D}$. Moreover, by [1, Theorem 2.1], we have rank $(I_{n-1} - B^*B) = 1$. These two together imply, by way of the singular value decomposition of B, that $\|B\| = 1$. Hence B is of class S_{n-1}. On the other hand, if it is the latter case, then $W(A)$ is also a circular disc centered at the origin. Therefore, $A = J_n$ by [5, Theorem 2.9]. This proves (b).

For (b) $\Rightarrow$ (c), if A is unitary (resp., $A = J_n$), then, obviously, $p_A = n_A = \infty$ (resp., $p_A = n_A = n - 1$). On the other hand, if A is unitarily similar to the asserted $[a\omega_n^j] \oplus B$, then $\|A\| = \max\{|a|, \|B\|\} = |a| = \rho(A)$. Thus $p_A = n_A = \infty$ by Proposition 1.2 (c) and 1.3.

Finally, for (c) $\Rightarrow$ (a), if $p_A = n_A = \infty$, then (a) is a consequence of Lemma 2.1. On the other hand, if $p_A = n_A = n - 1$, then $A^n = 0_n$. This implies that $A = J_n$ and thus (a) holds by Proposition 2.8. $\qquad\square$

The next theorem is a consequence of Theorem 2.5. It gives a lower bound, in terms of p_A, for $w(A)$ when A is an n-by-n matrix with $\|A\| = 1$.

Theorem 2.10. *If A is an n-by-n matrix with $\|A\| = \|A^k\| = 1$ for some $k \ge 1$, then $w(A) \ge \cos(\pi/(k + 2))$. Moreover, in this case, the following conditions are equivalent:*

(a) $w(A) = \cos(\pi/(k + 2))$,

(b) *A is unitarily similar to $J_{k+1} \oplus B$, where B is a finite matrix with $w(B) \le \cos(\pi/(k + 2))$, and*

(c) $W(A) = \{z \in \mathbb{C} : |z| \le \cos(\pi/(k + 2))\}$.

For the proof of (a) $\Rightarrow$ (b), we need the following lemma.

Lemma 2.11. *Let*

$$
A = \begin{bmatrix}
0 & a_1 & & & \\
& 0 & \ddots & & \\
& & \ddots & a_{n-2} & \\
& & & 0 & a_{n-1} \\
& & & & a
\end{bmatrix}
\quad and \quad
B = \begin{bmatrix}
0 & a_1 & & \\
& 0 & \ddots & \\
& & \ddots & a_{n-2} \\
& & & 0
\end{bmatrix}
$$

be n-by-n and $(n-1)$-by-$(n-1)$ matrices, respectively, where $n \geq 2$ and a_j is nonzero for all j. Then $w(A) > w(B)$.

Proof. We prove this by induction on n. If $n = 2$, then $A = \begin{bmatrix} 0 & a_1 \\ 0 & a \end{bmatrix}$ and $B = [0]$, in which case we obviously have $w(A) > 0 = w(B)$. Assume now that the assertion is true for the matrix A of size at most $n-1$ $(n \geq 3)$, and let A and B be of the above form. By considering $e^{i\theta}A$ for a suitable real θ instead of A, we may assume that $w(A)$ equals the largest eigenvalue of $\mathrm{Re}\,A$. Let

$$
C = \begin{bmatrix}
0 & a_1 & & \\
& 0 & \ddots & \\
& & \ddots & a_{n-3} \\
& & & 0
\end{bmatrix},
$$

and let $p(z)$, $q(z)$ and $r(z)$ be the characteristic polynomials of $\mathrm{Re}\,A$, $\mathrm{Re}\,B$ and $\mathrm{Re}\,C$, respectively. We expand the determinant of

$$
\begin{bmatrix}
z & -a_1/2 & & & \\
-\overline{a}_1/2 & z & \ddots & & \\
& \ddots & \ddots & \ddots & \\
& & \ddots & z & -a_{n-1}/2 \\
& & & -\overline{a}_{n-1}/2 & z - \mathrm{Re}\,a
\end{bmatrix}
$$

by minors on its last row to obtain $p(z) = (z - \mathrm{Re}\,a)q(z) - (|a_{n-1}|^2/4)r(z)$. Let α, β and γ be the largest eigenvalues of $\mathrm{Re}\,A$, $\mathrm{Re}\,B$ and $\mathrm{Re}\,C$, respectively. Then $\alpha = w(A)$, $\beta = w(B)$ and $\gamma = w(C)$. Since $\mathrm{Re}\,B$ (resp., $\mathrm{Re}\,C$) is a principal submatrix of $\mathrm{Re}\,A$ (resp., $\mathrm{Re}\,B$), we have $\beta \leq \alpha$ (resp., $\gamma \leq \beta$). Assume that

$\alpha = \beta$. Then the above equation yields

$$0 = p(\alpha) = (\alpha - \operatorname{Re} a)q(\beta) - \frac{1}{4}|a_{n-1}|^2 \gamma(\beta) = -\frac{1}{4}|a_{n-1}|^2 \gamma(\beta).$$

Since $a_{n-1} \neq 0$ and β is larger than or equal to all eigenvalues of $\operatorname{Re} C$, we infer from $\gamma(\beta) = 0$ that $\beta = \gamma$ or $w(B) = w(C)$. This contradicts our induction hypothesis for B and C. Hence we must have $\alpha > \beta$ or $w(A) > w(B)$. $\qquad\qquad\square$

Proof of Theorem 2.10. By Theorem 2.5, the assumption $\|A\| = \|A^k\| = 1$ implies that $w(A \otimes J_{k+1}) = w(J_{k+1})$. Hence

$$w(A) = \|J_{k+1}\| w(A) \geq w(A \otimes J_{k+1}) = w(J_{k+1}) = \cos \frac{\pi}{k+2}$$

as asserted.

We now prove the equivalence of (a), (b) and (c). The implications (b) $\Rightarrow$ (c) and (c) $\Rightarrow$ (a) are trivial. To prove (a) $\Rightarrow$ (b), let x be a unit vector in $\mathbb{C}^n$ such that $\|A^k x\| = 1$. Then $\|A^j x\| = 1$ for all j, $0 \leq j \leq k$. We now check that $A^{k+1} x = 0$. Assuming otherwise that $\|A^{k+1} x\| > 0$, let $u_t = [u_{t1} \ \ldots \ u_{t,k+2}]^T$ in $\mathbb{C}^{k+2} \otimes \mathbb{C}^n$, where

$$u_{tj} = \begin{cases} \dfrac{\sqrt{1-t^2}}{\|A^{k+1}x\|} A^{k+1} x & \text{if } j = 1, \\[2ex] t\sqrt{\dfrac{2}{k+2}} \sin \dfrac{(j-1)\pi}{k+2} A^{k-j+2} x & \text{if } j = 2, \ldots, k+2 \end{cases}$$

for any t, $0 < t < 1$. Note that

$$v \equiv \sqrt{\frac{2}{k+2}} \left[\sin \frac{\pi}{k+2} \quad \sin \frac{2\pi}{k+2} \quad \ldots \quad \sin \frac{(k+1)\pi}{k+2} \right]^T$$

is a unit vector in $\mathbb{C}^{k+1}$ with $\langle J_{k+1} v, v \rangle = \cos(\pi/(k+2))$ (cf. [8, Proposition 1 (3)]). Hence $\|u_t\| = ((1-t^2) + t^2 \|v\|^2)^{1/2} = 1$, and

$$\langle (J_{k+2} \otimes A) u_t, u_t \rangle = t\sqrt{1-t^2} \sqrt{\frac{2}{k+2}} \sin \frac{\pi}{k+2} \|A^{k+1} x\|$$

$$+ t^2 \frac{2}{k+2} \sum_{j=1}^k \sin \frac{j\pi}{k+2} \sin \frac{(j+1)\pi}{k+2} \|A^{k-j+1} x\|^2$$

$$= t\sqrt{1-t^2} \sqrt{\frac{2}{k+2}} \sin \frac{\pi}{k+2} \|A^{k+1} x\| + t^2 \langle J_{k+1} v, v \rangle$$

$$= t\sqrt{1-t^2} \sqrt{\frac{2}{k+2}} \sin \frac{\pi}{k+2} \|A^{k+1} x\| + t^2 \cos \frac{\pi}{k+2}.$$

13

To reach a contradiction, we need to find some t_0, $0 < t_0 < 1$, such that $\langle (J_{k+2} \otimes A)u_{t_0}, u_{t_0} \rangle > \cos(\pi/(k+2))$. This is the same as

$$t_0 \sqrt{1 - t_0^2} \sqrt{\frac{2}{k+2}} \sin \frac{\pi}{k+2} \|A^{k+1}x\| > (1 - t_0^2) \cos \frac{\pi}{k+2}$$

or

$$\frac{t_0}{\sqrt{1 - t_0^2}} > \sqrt{\frac{k+2}{2}} \frac{\cot \frac{\pi}{k+2}}{\|A^{k+1}x\|}.$$

Since $\lim_{t \to 1^-} t/\sqrt{1 - t^2} = \infty$, the existence of such a t_0 is guaranteed. On the other hand, we also have

$$\langle (J_{k+2} \otimes A)u_{t_0}, u_{t_0} \rangle \leq w(J_{k+2} \otimes A) \leq \|J_{k+2}\| w(A) = w(A) = \cos \frac{\pi}{k+2},$$

hence a contradiction. Thus we must have $A^{k+1}x = 0$. Let K be the subspace of $\mathbb{C}^n$ generated by $x, Ax, \ldots, A^k x$. Then $AK \subseteq K$. If A' is the restriction of A to K, then $A'^{k+1} = 0$ and $\|A'^j x\| = \|A^j x\| = 1$ for all j, $0 \leq j \leq k$. Hence $\|A'^j\| = 1$ for all such j's. Together with $A'^{k+1} = 0$, this says that $p_{A'} = k$ and thus $\dim K = k + 1$ by Proposition 1.2 (a). Therefore, A' is unitarily similar to a matrix of the form $[a_{ij}]_{i,j=1}^{k+1}$ with $a_{ij} = 0$ for all $i \geq j$. Since $1 = \|A'^k\| = |a_{12} \cdots a_{k,k+1}|$, we infer that $|a_{12}| = \cdots = |a_{k,k+1}| = 1$, and thus all the other a_{ij}'s are zero. Therefore, $[a_{ij}]_{i,j=1}^{k+1}$, and hence A', is unitarily similar to J_{k+1}. Then A is unitarily similar to a matrix of the form

$$\begin{bmatrix} J_{k+1} & \begin{array}{ccc} & 0 & \\ b_1 & \cdots & b_{n-k-1} \end{array} \\ \hline 0 & \begin{array}{ccc} c_1 & & * \\ & \ddots & \\ * & & c_{n-k-1} \end{array} \end{bmatrix}.$$

To show that all the b_j's are zero, we appeal to Lemma 2.11. Indeed, for each j, $1 \leq j \leq n - k - 1$, consider the $(k+2)$-by-$(k+2)$ matrix

$$A_j = \begin{bmatrix} J_{k+1} & \begin{array}{c} 0 \\ \vdots \\ 0 \\ b_j \end{array} \\ \hline 0 & c_j \end{bmatrix}.$$

If $b_j \neq 0$, then $w(A_j) > w(J_{k+1}) = \cos(\pi/(k+2))$ by Lemma 2.11, which contradicts

14

$w(A_j) \le w(A) = \cos(\pi/(k+2))$. This proves (a) $\Rightarrow$ (b). $\qquad\qquad\square$

Theorem 2.10 generalizes the classical result of Williams and Crimmins [17] for $k = 1$. The following corollary is for $k = n - 1$. Part of it has been proven in [19]: the equivalence of (b) and (c) is in [19, Theorem 1] and that of (b) and (d) in [19, p. 352].

Corollary 2.12. *The following conditions are equivalent for an n-by-n matrix A with $\|A\| = 1$:*

(a) $\|A^{n-1}\| = 1$ *and* $w(A) = \cos(\pi/(n+1))$,

(b) *A is unitarily similar to* J_n,

(c) $W(A) = \{z \in \mathbb{C} : |z| \le \cos(\pi/(n+1))\}$,

(d) $\|A^{n-1}\| = 1$ *and* $A^n = 0_n$, *and*

(e) $p_A = n_A = n - 1$.

Proof. The equivalence of (a) and (b) is by Theorem 2.10. The other implications are either in [19] or trivial. $\qquad\qquad\square$

Note that, in the preceding corollary, the conditions that $\|A\| = 1$ and $w(A) = \cos(\pi/(n+1))$ for an n-by-n matrix A are not sufficient to guarantee that A be unitarily similar to J_n. One example is $A = J_{n-1} \oplus [\cos(\pi/(n+1))]$.

We end this section with a characterization of matrices A satisfying $p_A = n_A$. This is related to the previous results.

Theorem 2.13. *Let A be an n-by-n matrix with $\|A\| = 1$. Then*

(a) *A satisfies $p_A = n_A$ ($\le \infty$) if and only if either* (i) *it has a unitary part, or* (ii) *it is unitarily similar to a direct sum $J_{k+1} \oplus B$, where $k = p_A < \infty$ and $B^{k+1} = 0_{n-k-1}$, and*

(b) *if $p_A = n_A$ ($\le \infty$), then $w(A \otimes A) = w(A)$ holds, but not conversely.*

Proof. (a) For the necessity, we may assume, in view of Proposition 1.2 (c), that $k \equiv p_A = n_A < \infty$ and prove that A is unitarily similar to the asserted direct sum. Since $A^{k+1} = 0_n$, A is unitarily similar to a block matrix A' of the form $[A_{ij}]_{i,j=1}^{k+1}$

15

with $A_{ij} = 0$ for $1 \leq j \leq i \leq k+1$. Hence

$$A'^k = \begin{bmatrix} 0 \cdots & 0 & \prod_{i=1}^{k} A_{i,i+1} \\ & 0 & 0 \\ & \ddots & \vdots \\ & & 0 \end{bmatrix}.$$

Since $\|A'^k\| = \|A^k\| = \|A\|^k = 1$, we have $\|\prod_{i=1}^{k} A_{i,i+1}\| = 1$. Let x be a unit vector such that $\|(\prod_{i=1}^{k} A_{i,i+1})x\| = 1$. Then $\|(\prod_{i=j}^{k} A_{i,i+1})x\| = 1$ for all j, $1 \leq j \leq k$. Let $\{e_1, \ldots, e_{k+1}\}$ be the standard basis for $\mathbb{C}^{k+1}$, and let $x_j = e_j \otimes (\prod_{i=j}^{k} A_{i,i+1})x$ if $1 \leq j \leq k$, and $x_{k+1} = e_{k+1} \otimes x$. Then $x_1, \ldots, x_{k+1}$ are orthonormal vectors in $\mathbb{C}^n$, and $A'x_1 = 0$ and $A'x_j = x_{j-1}$ for $2 \leq j \leq k+1$. Thus if K is the subspace generated by $x_1, \ldots, x_{k+1}$, then $\dim K = k+1$, $A'K \subseteq K$, and the restriction of A' to K is unitarily similar to J_{k+1}. We infer from $\|A'\| = 1$ and $A'^* x_{k+1} = 0$ that K reduces A', and thus A' is unitarily similar to $J_{k+1} \oplus B$ with $B^{k+1} = 0$.

For the converse, if A has a unitary part, then $p_A = n_A = \infty$ by Proposition 1.2 (c). On the other hand, if A is unitarily similar to $J_{k+1} \oplus B$ with the asserted properties, then $A^{k+1} = 0$ implies that $p_A \leq n_A \leq k$. But

$$\|A^k\| = \|J_{k+1}^k \oplus B^k\| = \max\{\|J_{k+1}^k\|, \|B^k\|\} = 1 = \|A\|^k$$

and $\|A^{k+1}\| = 0 < 1 = \|A\|^{k+1}$ together yield $p_A = n_A = k$.

(b) If A has a unitary part, then $w(A \otimes A) = w(A)$ by Proposition 2.1. On the other hand, if A is unitarily similar to $J_{k+1} \oplus B$ as in (a), then $A \otimes A$ is unitarily similar to $(J_{k+1} \otimes J_{k+1}) \oplus (J_{k+1} \otimes B) \oplus (B \otimes J_{k+1}) \oplus (B \otimes B)$. Note that $w(J_{k+1} \otimes J_{k+1}) = w(J_{k+1})$ by Proposition 2.8, and

$$w(J_{k+1} \otimes B) = w(B \otimes J_{k+1}) \leq \|J_{k+1}\| w(B) = w(B) \tag{1}$$

by Proposition 1.1. Since $B^{k+1} = 0$ and $\|B\| \leq 1$, [21, Lemma 3 (a)] implies that B can be dilated to the direct sum of $\text{rank}(I - B^*B)$ copies of J_m for some $m \leq k+1$. Thus $w(B) \leq w(J_m) \leq w(J_{k+1})$. Combined with (1), this yields $w(J_{k+1} \otimes B) \leq w(J_{k+1})$. Also,

$$w(B \otimes B) \leq \|B\| w(B) \leq w(B) \leq w(J_{k+1}).$$

Therefore,

$$w(A \otimes B) = \max\{w(J_{k+1} \otimes J_{k+1}), w(J_{k+1} \otimes B), w(B \otimes B)\}$$

$$= w(J_{k+1})$$

$$= \max\{w(J_{k+1}), w(B)\}$$

$$= w(A).$$

That $w(A \otimes A) = w(A)$ does not imply $p_A = n_A$ is seen by $A = J_2 \oplus [a]$, where $0 < |a| \le 1/2$, in which case, $\|A\| = 1$ and $w(A \otimes A) = w(A) = 1/2$, but $p_A = 1$ and $n_A = \infty$. $\qquad\square$

The final result of this section is conditions for a matrix A with $p_A = n_A$ so that it be unitarily similar to a block-shift matrix

$$A' = \begin{bmatrix} 0 & A_1 & & \\ & 0 & \ddots & \\ & & \ddots & A_k \\ & & & 0 \end{bmatrix} \tag{2}$$

with $\|A_1 \cdots A_k\| = \|A\|$.

Proposition 2.14. *Let A be an n-by-n matrix with $p_A = n_A \equiv k < \infty$. If either (a) $k = 1$, $n - 2$ or $n - 1$, or (b) $n = 2, 3, 4$ or 5, then A is unitarily similar to the block-shift matrix A' in (2) with $\|A_1 \cdots A_k\| = \|A\|$.*

Proof. We may assume that $\|A\| = 1$.

(a) If $k = n_A = 1$, then $A^2 = 0_n$. Hence A is unitarily similar to a block-shift matrix of the form $\begin{bmatrix} 0 & A_1 \\ 0 & 0 \end{bmatrix}$ with $\|A_1\| = \|A\|$.

If $k = p_A = n_A = n - 1$ (resp., $n - 2$), then Theorem 2.13 (a) implies that A is unitarily similar to J_n (resp., $J_{n-1} \oplus [0]$). The latter matrix plays the role of A' with $k = n - 1$ (resp., $n - 2$) and $A_1 = \cdots = A_{n-1} = [1]$ (resp., $A_1 = \cdots = A_{n-3} = [1]$ and $A_{n-2} = [1 \ 0]$).

(b) In light of (a), we need only prove for $n = 5$ and $k = 2$. Invoking Theorem 2.13 to obtain the unitary similarity of A and $J_3 \oplus \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix}$, where $|b| \le 1$. The latter matrix is permutationally similar to a block-shift matrix A' with $k = 2$, $A_1 = \begin{bmatrix} 1 & 0 \\ 0 & b \end{bmatrix}$

and $A_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. We obviously have $\|A_1 A_2\| = \|\begin{bmatrix} 1 \\ 0 \end{bmatrix}\| = 1 = \|A\|$. $\square$

We remark that the preceding proposition fails for $n = 6$ and $k = 2$. Here is an example. Let $A = J_3 \oplus B$, where

$$B = b \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

with $b = \sqrt{2/(3 + \sqrt{5})}$. Then $\|A^2\| = 1 = \|A\|^2$ and $A^3 = 0_6$. This shows that $p_A = n_A = 2$. Since $w(B) = 2b > \sqrt{2}/2 = w(J_3)$ and $w(B)$ is not a circular disc centered at the origin (cf. [13, Theorem 4.1 (2)]), we infer that nor is $W(A)$ (= the convex hull of $W(J_3) \cup W(B)$). This implies that A cannot be unitarily similar to a block-shift matrix.

3.   Nonnegative Matrices

Recall that a matrix $A = [a_{ij}]_{i,j=1}^n$ is *nonnegative* (resp., *positive*), denoted by $A \succcurlyeq 0$ (resp., $A \succ 0$), if $a_{ij} \geq 0$ (resp., $a_{ij} > 0$) for all i and j. Two n-by-n matrices A and B are *permutationally similar* if there is an n-by-n permutation matrix P (one with each row and column has exactly one 1 and all other entries 0) such that $P^T A P = B$. A is said to be (*permutationally*) *reducible* if either A is the 1-by-1 zero matrix or $n \geq 2$ and it is permutationally similar to a matrix of the form $\begin{bmatrix} B & C \\ 0 & D \end{bmatrix}$, where B and D are square matrices; otherwise, it is (*permutationally*) *irreducible*. It is known that if A is nonnegative with $\mathrm{Re}\,A$ irreducible, then it is permutationally similar to a block-shift matrix if and only if its numerical range is a circular disc centered at the origin (cf. [16, Theorem 1 (a)⇔(r)]). Other properties of nonnegative matrices can be found in [11, Section 6.2 and Chapter 8].

The main result of this section is the following theorem, which essentially generalizes Theorem 2.5.

Theorem 3.1. *Let A be an n-by-n matrix and B an m-by-m nonnegative matrix with $\mathrm{Re}\,B$ irreducible. Then the following conditions are equivalent:*

(a) $w(A \otimes B) = \|A\| w(B)$,

(b) *either* (i) $p_A = \infty$, *or* (ii) $n_B \leq p_A < \infty$ *and $W(B)$ is a circular disc centered at the origin, and*

(c) *either* (i) $p_A = \infty$, *or* (ii) $n_B \leq p_A < \infty$ *and B is permutationally similar to a block-shift matrix*

$$\begin{bmatrix} 0 & B_1 & & \\ & 0 & \ddots & \\ & & \ddots & B_k \\ & & & 0 \end{bmatrix}$$

with $k = n_B$.

For its proof, we need the following two lemmas.

Lemma 3.2. *Let $A = [a_{ij}]_{i,j=1}^n$ be a nonnegative matrix. Then the following hold:*

(a) *The index n_A is finite if and only if there is no sequence of indices $i_0, i_1, \ldots, i_{k-1}, i_k$ ($k \geq 1$) with $i_0 = i_k$ such that $a_{i_0 i_1}, \ldots, a_{i_{k-1} i_k}$ are all nonzero. In particular, we have $n_A = \sup\{k \geq 1 :$ there are distinct $i_j, 0 \leq j \leq k,$ such that $a_{i_j i_{j+1}} \neq 0$ for all $j\}$.*

(b) *$n_A = \infty$ if and only if there is a $k \geq 1$ such that some diagonal entry of A^k is nonzero.*

(c) *If $a_{ii} \neq 0$ for some i, $1 \leq i \leq n$, then $n_A = \infty$.*

(d) *If A is irreducible, then $n_A = \infty$.*

(e) *If A is the block-shift matrix*

$$\begin{bmatrix} 0_{n_1} & A_1 & & \\ & 0_{n_2} & \ddots & \\ & & \ddots & A_k \\ & & & 0_{n_{k+1}} \end{bmatrix} \quad on \quad \mathbb{C}^n = \mathbb{C}^{n_1} \oplus \cdots \oplus \mathbb{C}^{n_{k+1}}$$

and $\operatorname{Re} A$ is irreducible, then $k = n_A$.

Proof. (a) Assume first that the indices $i_0, i_1, \ldots, i_{k-1}, i_k = i_0$ ($k \geq 1$) are such that $a_{i_0 i_1}, \ldots, a_{i_{k-1} i_k} \neq 0$. [11, Theorem 6.2.16] says that this is the case if and only if $(A^k)_{i_0 i_0}$, the (i_0, i_0)-entry of A^k, is nonzero. Hence $A^k \neq 0_n$. Similarly, considering the sequence $i_0, \ldots, i_k, i_1, \ldots, i_k, \ldots, i_1, \ldots, i_k$ of $\ell k + 1$ indices for any $\ell \geq 1$, we also obtain $A^{\ell k} \neq 0_n$. It follows that $n_A = \infty$. Conversely, assume that $n_A = \infty$. Then $A^k \neq 0_n$ for some $k \geq n$. [11, Theorem 6.2.16] yields that, for some i and j, there are indices $i_0 = i, i_1, \ldots, i_{k-1}, i_k = j$ such that $a_{i_0 i_1}, \ldots, a_{i_{k-1} i_k}$ are all nonzero. By the pigeonhole principle, we infer that $i_s = i_t$ for some s and t, $0 \leq s < t \leq k$. Then $i_s, \ldots, i_t$ are such that $i_s = i_t$ and $a_{i_s i_{s+1}}, \ldots, a_{i_{t-1} i_t} \neq 0$. This

19

proves the converse. The expression for n_A is an easy consequence of [11, Theorem 6.2.16] and the above arguments. So are (b) and (c).

(d) Note that the irreducibility of A is equivalent to the existence, for every distinct pair i and j, of indices $i_0 = i, i_1, \ldots, i_{k-1}, i_k = j$ $(k \geq 1)$ such that $a_{i_0 i_1}, \ldots, a_{i_{k-1} i_k}$ are all nonzero. Combining such indices from i to j with those from j to i yields one from i to i with the corresponding entries nonzero. Thus $n_A = \infty$ by [11, Theorem 6.2.16] and (b).

(e) Since $A^{k+1} = 0_n$, we have $n_A \leq k$. If $n_A < k$, then $A^k = 0_n$, which implies that $A_1 \cdots A_k = 0$. If there are any nonzero $a_{i_0 i_1}, a_{i_1 i_2}, \ldots, a_{i_{k-1} i_k}$, where $(\sum_{j=1}^{\ell} n_j) + 1 \leq i_\ell \leq \sum_{j=1}^{\ell+1} n_j$ for $0 \leq \ell \leq k$, then the $(i_0, n_{k+1} - (n - i_k))$-entry of $A_1 \cdots A_k$, being larger than or equal to $\prod_{j=0}^{k-1} a_{i_j i_{j+1}}$, is nonzero, which contradicts the zeroness of the product $A_1 \cdots A_k$. Thus no such nonzero sequence exists. This results in the reducibility of $\operatorname{Re} A$, a contradiction. Hence we must have $n_A = k$. $\qquad \square$

We remark that the conditions in the preceding lemma can all be expressed equivalently in terms of the directed graph associated with the matrix A (cf. [11, Section 6.2]).

Lemma 3.3. *Let A and B be n-by-n and m-by-m matrices, respectively. If B is unitarily similar to a block-shift matrix*

$$
\begin{bmatrix}
0_{m_1} & B_1 & & \\
& 0_{m_2} & \ddots & \\
& & \ddots & B_k \\
& & & 0_{m_{k+1}}
\end{bmatrix}
\quad on \ \mathbb{C}^m = \mathbb{C}^{m_1} \oplus \cdots \oplus \mathbb{C}^{m_{k+1}}
\tag{3}
$$

with $k \leq p_A \leq \infty$, then $w(A \otimes B) = \|A\| w(B)$.

Proof. We may assume that $\|A\| = 1$ and B is equal to the block-shift matrix (3). Since $k \leq p_A \leq \infty$, we have $\|A^k\| = \|A\|^k = 1$. Let x be a unit vector in $\mathbb{C}^n$ such that $\|A^k x\| = 1$, and let $y = [y_1 \ \ldots \ y_{k+1}]^T$, where y_j is in $\mathbb{C}^{m_j}$, $1 \leq j \leq k+1$, be a unit vector in $\mathbb{C}^m$ such that $|\langle By, y \rangle| = w(B)$. Let $u = [y_1 \otimes A^k x \ \ y_2 \otimes A^{k-1} x \ \ \ldots \ \ y_{k+1} \otimes x]^T$. Then u is a vector in $\mathbb{C}^m \otimes \mathbb{C}^n$ with

$$
\|u\| = \left(\sum_{j=1}^{k+1} \|y_j \otimes A^{k-j+1} x\|^2 \right)^{1/2} = \left(\sum_{j=1}^{k+1} \|y_j\|^2 \|A^{k-j+1} x\|^2 \right)^{1/2}
$$

$$
= \left(\sum_{j=1}^{k+1} \|y_j\|^2 \right)^{1/2} = \|y\| = 1.
$$

Moreover, we have

$$
|\langle (B \otimes A)u, u \rangle|
$$

$$
= \left| \left\langle \begin{bmatrix} 0_{m_1 n} & B_1 \otimes A & & \\ & 0_{m_2 n} & \ddots & \\ & & \ddots & B_k \otimes A \\ & & & 0_{m_{k+1} n} \end{bmatrix} \begin{bmatrix} y_1 \otimes A^k x \\ y_2 \otimes A^{k-1} x \\ \vdots \\ y_{k+1} \otimes x \end{bmatrix}, \begin{bmatrix} y_1 \otimes A^k x \\ y_2 \otimes A^{k-1} x \\ \vdots \\ y_{k+1} \otimes x \end{bmatrix} \right\rangle \right|
$$

$$
= \left| \sum_{j=1}^{k} \langle (B_j y_{j+1}) \otimes (A^{k-j+1} x), y_j \otimes (A^{k-j+1} x) \rangle \right|
$$

$$
= \left| \sum_{j=1}^{k} \langle B_j y_{j+1}, y_j \rangle \| A^{k-j+1} x \|^2 \right|
$$

$$
= \left| \sum_{j=1}^{k} \langle B_j y_{j+1}, y_j \rangle \right|
$$

$$
= |\langle By, y \rangle| = w(B).
$$

This shows that $w(B) \leq w(B \otimes A) = w(A \otimes B)$. But $w(A \otimes B) \leq \|A\| w(B) = w(B)$ always holds by Proposition 1.1. Hence $w(A \otimes B) = w(B)$ as asserted. $\square$

We are now ready to prove Theorem 3.1.

Proof of Theorem 3.1. For (a) $\Rightarrow$ (b), We assume that $\|A\| = 1$ and A is c.n.u. In view of Theorem 2.2 and Proposition 1.2 (c), we need only check that $w(A \otimes B) = w(B)$ implies $n_B \leq p_A$ ($< \infty$). Let $B = [b_{ij}]_{i,j=1}^{m}$, and let x be a unit vector in $\mathbb{C}^m \otimes \mathbb{C}^n$ such that $w(B \otimes A) = |\langle (B \otimes A)x, x \rangle|$. If $x = [x_1 \ \ldots \ x_m]^T$, where x_j is in $\mathbb{C}^n$ for $1 \leq j \leq m$, then

$$
w(B) = w(B \otimes A) = |\langle [b_{ij} A] x, x \rangle|
$$

$$
\leq \sum_{i,j} b_{ij} |\langle A x_j, x_i \rangle|
$$

$$
\leq \sum_{i,j} b_{ij} \| A x_j \| \| x_i \| \tag{4}
$$

$$
\leq \|A\| \sum_{i,j} b_{ij} \| x_j \| \| x_i \| \tag{5}
$$

$$
\leq \langle B x', x' \rangle
$$

$$
\leq w(B), \tag{6}
$$

where $x' = [\|x_1\| \; \ldots \; \|x_m\|]^T$ is a unit vector in $\mathbb{C}^m$. This shows that the above inequalities are equalities throughout. Since $B \succcurlyeq 0$ and $\mathrm{Re}\,B$ is irreducible, there is a unique unit vector y in $\mathbb{C}^m$ with $y \succ 0$ such that $\langle By, y \rangle = w(B)$ (cf. [14, Proposition 3.3]). The equality in (6) yields that $x' = y$ and thus $x_j \neq 0$ for all j. Also, the equalities in (4) and (5) imply that $|\langle Ax_j, x_i \rangle| = \|Ax_j\|\|x_i\| = \|x_j\|\|x_i\|$ for all those b_{ij}'s with $b_{ij} > 0$. Thus $Ax_j = \lambda_{ij} x_i$ for some λ_{ij} satisfying $|\lambda_{ij}| = \|x_j\|/\|x_i\|$. Assume first that $k \equiv n_B < \infty$. Thus $B^k \neq 0_m$. By Lemma 3.2 (a), there are distinct indices $i_0, \ldots, i_k$ such that $b_{i_0 i_1}, \ldots, b_{i_{k-1} i_k} > 0$. It thus follows from above that $Ax_{i_j} = \lambda_{i_{j-1} i_j} x_{i_{j-1}}$ for $1 \leq j \leq k$. Hence $A^k x_{i_k} = (\prod_{j=1}^k \lambda_{i_{j-1} i_j}) x_{i_0}$. Since

$$\|A^k x_{i_k}\| = (\prod_{j=1}^k \frac{\|x_{i_j}\|}{\|x_{i_{j-1}}\|})\|x_{i_0}\| = \|x_{i_k}\|,$$

we obtain $\|A^k\| = 1$ or $p_A \geq k = n_B$. On the other hand, if $n_B = \infty$, then the same arguments as above with k arbitrarily large yield that $p_A = \infty$, which contradicts our assumption that A is c.n.u. This proves (a) $\Rightarrow$ (b).

That (b) $\Leftrightarrow$ (c) is a consequence of [16, Theorem 1 (a)$\Leftrightarrow$(r)], and (c) $\Rightarrow$ (a) is by Lemma 3.2 (e) and Lemma 3.3. $\square$

Note that, in Theorem 3.1, the implication (a) $\Rightarrow$ (b) or (a) $\Rightarrow$ (c) is no longer true if B is nonnegative but without the irreducibility of $\mathrm{Re}\,B$. One example is $A = B = J_2 \oplus [a]$, where $0 < a \leq 1/2$ (cf. the end of the proof of Theorem 2.13 (b)). The next example shows that the same can be said if B is not nonnegative but $\mathrm{Re}\,B$ is irreducible.

Example 3.4. Let $A = J_3$ and

$$B = \begin{bmatrix} 0 & -\sqrt{2} & 1 \\ 0 & 0 & 1 \\ 0 & 0 & \sqrt{2}/2 \end{bmatrix}.$$

Then $\mathrm{Re}\,B$ is easily seen to be irreducible. We now show that $W(B) = \overline{\mathbb{D}}$. This is seen via [13, Corollary 2.5] by letting $u = 0$ and $\lambda = \sqrt{2}/2$ therein and checking that

$$\mathrm{tr}\,(B^* B^2) = \mathrm{tr} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & \sqrt{2}/4 \end{bmatrix} = \frac{\sqrt{2}}{4} = \lambda |\lambda|^2$$

and $\operatorname{tr}(B^*B) = 9/2 \geq 5|\lambda|^2$, where $\operatorname{tr}(\cdot)$ denotes the trace of a matrix. We next prove that 1 is an eigenvalue of $\operatorname{Re}(A \otimes B)$. Indeed, since

$$\operatorname{Re}(A \otimes B) = \frac{1}{2} \begin{bmatrix} 0_3 & B & 0_3 \\ B^* & 0_3 & B \\ 0_3 & B^* & 0_3 \end{bmatrix},$$

we need to check that

$$\det \begin{bmatrix} 2I_3 & -B & 0_3 \\ -B^* & 2I_3 & -B \\ 0_3 & -B^* & 2I_3 \end{bmatrix} = 0.$$

By a repeated use of the Schur decomposition, the above determinant is seen to be equal to

$$\det(2I_3) \det \left(\begin{bmatrix} 2I_3 & -B \\ -B^* & 2I_3 \end{bmatrix} - \begin{bmatrix} -B^* \\ 0_3 \end{bmatrix} (\frac{1}{2}I_3) \begin{bmatrix} -B & 0_3 \end{bmatrix} \right)$$

$$= 8 \det \begin{bmatrix} 2I_3 - (1/2)B^*B & -B \\ -B^* & 2I_3 \end{bmatrix}$$

$$= 8 \det(4I_3 - B^*B - BB^*)$$

$$= 8 \det \begin{bmatrix} 1 & -1 & -\sqrt{2}/2 \\ -1 & 1 & \sqrt{2}/2 \\ -\sqrt{2}/2 & \sqrt{2}/2 & 1 \end{bmatrix}$$

$$= 0$$

as required. Since $W(A \otimes B)$ is a circular disc centered at the origin (by the unitary similarity of $A \otimes B$ and $e^{i\theta}(A \otimes B)$ for all real θ) and $w(A \otimes B) \leq \|A\|w(B) = 1$, we infer from $1 \in \sigma(\operatorname{Re}(A \otimes B))$ that $W(A \otimes B) = \overline{\mathbb{D}}$. Hence $w(A \otimes B) = 1 = \|A\|w(B)$. But, obviously, we have $n_B = \infty$ and $p_A = 2$. $\qquad\square$

The next corollary gives a more concrete equivalent condition, in terms of block-shift matrices, for $w(A \otimes B) = \|A\|w(B)$ when $A = B \succcurlyeq 0$ and $\operatorname{Re} B$ is irreducible.

Corollary 3.5. *Let A be an n-by-n nonnegative matrix with $\operatorname{Re} A$ irreducible. Then the following conditions are equivalent:*

(a) $w(A \otimes A) = \|A\|w(A)$,

(b) $p_A = n_A \ (\leq \infty)$, *and*

(c) *either A is unitarily similar to $[a] \oplus A'$ with $|a| \geq \|A'\|$, or A is permutationally*

similar to a block-shift matrix

$$A'' = \begin{bmatrix} 0 & A_1 & & & \\ & 0 & \ddots & & \\ & & & \ddots & A_k \\ & & & & 0 \end{bmatrix}$$

with $\|A_1 \cdots A_k\| = \|A\|$.

Proof. We may assume that $\|A\| = 1$. The implication (a) $\Rightarrow$ (b) is by Theorem 3.1 and Proposition 1.3 (d). For (b) $\Rightarrow$ (c), if $p_A = n_A = \infty$, then A has a unitary part by Proposition 1.2 (c), and hence A is unitarily similar to $[a] \oplus A'$ with $|a| = 1 \geq \|A'\|$ as asserted. On the other hand, if $p_A = n_A < \infty$, then $w(A \otimes A) = w(A)$ by Theorem 2.13 (b). Hence Theorem 2.2 implies that $W(A)$ is a circular disc centered at the origin. For a nonnegative A with $\operatorname{Re} A$ irreducible, this is equivalent to A being permutationally similar to the block-shift matrix A'' (cf. [16, Theorem 1 (a)$\Leftrightarrow$(r)]). As $n_{A''} = k$ by Lemma 3.2 (e), we also have $p_A = k$. Thus $\|A^k\| = \|A\|^k = 1$, which yields that $\|A_1 \cdots A_k\| = 1 = \|A\|$ as required. Finally, for (c) $\Rightarrow$ (a), if A is unitarily similar to $[a] \oplus A'$ with $|a| \geq \|A'\|$, then $w(A \otimes A) = w(A)$ by Lemma 2.1. On the other hand, if A is permutationally similar to the block-shift matrix A'' with $\|A_1 \cdots A_k\| = 1$, then

$$\|A^k\| = \|A''^k\| = \|A_1 \cdots A_k\| = 1 = \|A\|^k.$$

Thus $p_A \geq k = n_A$. The equality $w(A \otimes A) = w(A)$ then follows from Theorem 3.1. $\qquad\square$

Corollary 3.6. *Let* $A = [a_{ij}]_{i,j=1}^n$, *where* $a_{ij} \geq 0$ *for all* i *and* j, $a_{ij} = 0$ *for* $i \geq j$, *and* $a_{i,i+1} > 0$ *for all* i. *Then the following conditions are equivalent:*

(a) $w(A \otimes A) = \|A\|w(A)$,

(b) $p_A = n_A = n - 1$, *and*

(c) $a_{12} = \cdots = a_{n-1,n}$ *and* $a_{ij} = 0$ *for all other pairs of* i *and* j.

Proof. In this case, A is nonnegative, $\operatorname{Re} A$ is irreducible and $n_A = n - 1$. Consequently, Corollary 3.5 yields the equivalence of (a), (b) and the condition (c') that A is permutationally similar to a block-shift matrix A'' as in Corollary 3.5 (c). Since $k = n_{A''} = n_A$ by Lemma 3.2 (e), A'' is necessarily equal to A with $|a_{12} \cdots a_{n-1,n}| = \|A\|$ and $a_{ij} = 0$ for all other pairs of i and j. The norm condition

above yields that $a_{12} = \cdots = a_{n-1,n} = \|A\|$. Thus (c') is the same as (c), and we have the equivalence of (a), (b) and (c). $\square$

Acknowledgements

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參加「第四屆矩陣分析及其應用國際會議」報告

國立交通大學應用數學系

吳培元

　　本次會議是在 2013 年 7 月 2 日至 5 日在 Konya,Turkey 舉行。我是在 6 月 25 日自桃園出發，搭機經香港，轉機土耳其航空公司，於次日清晨才抵達伊斯坦堡，在此地旅遊觀光。住了五夜後，再於 7 月 1 日搭機一個小時到開會地點的 Konya Dedeman Hotel 入住報到。

　　正式會議由 7 月 2 日開始進行。此項會議，目前已進行到第四屆。由第零屆起，每隔一、兩年或三年舉行一次，已分別在 Fort Lauderdale, Florida, USA, Beijing, China, LinAn, Zhejiang, China 等地舉辦過。這一次是首次在土耳其舉行，由 Selcuk University 協辦。會議開始後，先由現任的「國際線性代數學會」主席 Steve Kirkland 演講〝Spectral graph theory in action〞，他以「平行計算」和「量子計算」的兩個例子來說明譜圖論的應用，說理清楚，得到與會者的高度肯定，其後還有 S. Garcia, M.S. Moslehian 和 H.J. Woerdeman 每人五十分鐘的演講，表現也都在水準之上。

　　中餐過後，下午兩點起，演講分四組平行進行。我擔任其中一組的主席，一個下午下來，共主持了十一場每人二十分鐘的演講，中間只有一段二十分鐘的休息時間。演講者多為土耳其本地的學者，但也

包括兩位原籍大陸的華人:上海大學的王卿文和加拿大 Univ. of Manitoba 的張揚。

次日,7 月 3 日的會議安排,上午有四場各五十分鐘的演講,其中最知名的是 Alexander Klyachko,作一場 ILAS Lecture 的演講〝Quantum margins and related matrix spectral problems〞,他原籍俄國,現任教於土耳其的 Bilkent Univ.,專長領域是光學、原子物理及放射線等,他在二十世紀末解決了一個久懸的數學問題:三個 Hermitian matrices A,B 和 A+B 的特徵值所有的不等式關係,並和 Schubert calculus for the homology of Grassmannians 及 moduli 的 vector bundles on P^2 作連結,因而在數學界聲名大噪。只是他的演講材料太多且牽涉到物理的概念,很難跟得上。其餘演講者包括譚天佑和施能聖,則是每年開會遇見的常客。下午仍是四組演講平行進行,我只聽了一場會議主辦人張福振(Nova Southeastern Univ.)的〝The Schur complement〞,是他在最近出版的一本書中的一章的內容概要。當天晚上有一場會議主辦單位的晚宴活動。

7 月 4 日是旅遊活動時間。與會者五十餘人參加前往 Cappadocia 的自費一日旅遊。

7 月 5 日上午是兩天半會議的最後半天,共有四場五十分鐘的演講。包括了儲德林(新加坡國立大學),張曉東(上海交通大學)和本人,

我的演講是最後一個，講題:Numerical radii of tensor products。反應尚可，有些與會者都已先離去，在場者並沒有作數值域研究的專家。兩天半的會議隨即結束。當天下午則由大會當局安排作 Konya 的市內旅遊和參觀。

本次會議承蒙土耳其主辦單位熱情款待，特此表達感謝之意。也感謝國科會研究計畫內的旅費及住宿的支助。

Numerical Radii of Tensor Products

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A $n \times n$ complex matrix

(1) $W(A) = \{\langle Ax, x \rangle : x \in \mathbb{C}^n, \|x\| = 1\}$: numerical range of A

$\quad x = [x_1 ... x_n]^T, \ y = [y_1 ... y_n]^T$

$\Rightarrow \langle x, y \rangle = x_1 \overline{y_1} + \cdots + x_n \overline{y_n}$

$\quad \|x\| = (|x_1|^2 + \cdots + |x_n|^2)^{1/2}$

(2) $w(A) = \max\{|z| : z \in W(A)\}$: numerical radius of A

Properties:

(1) $W(A)$ nonempty compact subset of $\mathbb{C}$

(2) U unitary $\Rightarrow W(U^*AU) = W(A)$

(3) $A \cong \begin{bmatrix} B & * \\ * & * \end{bmatrix} \Rightarrow W(B) \subseteq W(A)$

(4) $\lambda \in W(A) \Leftrightarrow A \cong \begin{bmatrix} \lambda & * \\ * & * \end{bmatrix}$

(5) Toeplitz–Hausdorff (1918–19):

$W(A)$ convex

(6) $\sigma(A) \subseteq W(A)$

(7) $W(A \oplus B) = (W(A) \cup W(B))^{\curvearrowright}$

(8) A normal $\Rightarrow W(A) = \sigma(A)^{\curvearrowright}$

(9) $\|A\|/2 \le w(A) \le \|A\|$ $(\equiv \max\{\|Ax\| : x \in \mathbb{C}^n, \|x\| = 1\})$

$$A = [a_{ij}]_{i,j=1}^n, \quad B = [b_{ij}]_{i,j=1}^m$$

$$A \otimes B = [a_{ij}B]_{i,j=1}^n \ (mn) \times (mn)$$

(1) $A_1 \cong A_2 \Rightarrow A_1 \otimes B \cong A_2 \otimes B$

(2) $A \otimes B \cong B \otimes A$

(3) $\|A \otimes B\| = \|A\| \|B\|$

(4) $W(A \otimes B) \supseteq W(A)W(B)$

Pf.: Let $\lambda \in W(A)$

$$\Rightarrow A \cong \begin{bmatrix} \lambda & * \\ * & * \end{bmatrix}$$

$$\Rightarrow A \otimes B \cong \begin{bmatrix} \lambda B & * \\ * & * \end{bmatrix}$$

$$\Rightarrow W(A \otimes B) \supseteq \lambda W(B) \quad \forall \lambda \in W(A)$$

$$\Rightarrow W(A \otimes B) \supseteq W(A)W(B)$$

(5) $w(A \otimes B) \geq w(A)w(B)$

(6) A normal $\Rightarrow W(A \otimes B) = (W(A)W(B))\hat{}$

Pf.: $\because A \cong \text{diag } (a_1, ..., a_n)$

$$\Rightarrow W(A \otimes B) = W(\begin{bmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{bmatrix} \otimes B) = W(a_1 B \oplus \cdots \oplus a_n B)$$

$$= (\cup_j W(a_j B))\hat{}$$

$$= (W(A)W(B))\hat{}$$

(7) A normal $\Rightarrow w(A \otimes B) = w(A)w(B) = \|A\|w(B)$

(8) $w(A \otimes B) \leq \|A\|w(B)$

Pf.: May assume $\|A\| = 1$

$\because U = \begin{bmatrix} A & (I_n - AA^*)^{1/2} \\ (I_n - A^*A)^{1/2} & -A^* \end{bmatrix}$ unitary (Halmos, 1950)

$\Rightarrow w(A \otimes B) \leq w(U \otimes B) = \|U\|w(B) = \|A\|w(B)$

Questions:

(a) When $w(A \otimes B) = w(A)w(B)$?

(b) When $w(A \otimes B) = \|A\|w(B)$?

$$\text{(a) } w(A \otimes B) = w(A)w(B)$$

Prop. $w(A \otimes B) = w(A)w(B) \Rightarrow w(A) = \rho(A)$ or $w(B) = \rho(B)$

Def. $\rho(A) = \max\{|\lambda| : \lambda \in \sigma(A)\}$

Pf.: Let $a \in W(A) \ni |a| = w(A)$

Let $b \in W(B) \ni |b| = w(B)$

May assume $a = b = 1$

$\therefore A \cong \begin{bmatrix} 1 & u \\ -u^* & * \end{bmatrix}, B \cong \begin{bmatrix} 1 & v \\ -v^* & * \end{bmatrix}$ (by Lma below)

$\Rightarrow A \otimes B \cong \begin{bmatrix} 1 & u \otimes v \\ u^* \otimes v^* & * \end{bmatrix}$

$\because w(A \otimes B) = w(A)w(B) = 1$

$\Rightarrow u^* \otimes v^* = -(u \otimes v)^* = -(u^* \otimes v^*)$

$\Rightarrow u^* \otimes v^* = 0$

$\Rightarrow u^* = 0$ or $v^* = 0$

$\Rightarrow 1 \in \sigma(A)$ or $1 \in \sigma(B)$

$\Rightarrow w(A) = \rho(A)$ or $w(B) = \rho(B)$

Lma. $A \cong \begin{bmatrix} w(A)e^{i\theta} & u \\ v & * \end{bmatrix} \Rightarrow v = -u^* e^{2i\theta}$

Note. $w(A) = \rho(A)$ & $w(B) = \rho(B) \nRightarrow w(A \otimes B) = w(A)w(B)$

$$\text{Ex. } A = \left[\begin{array}{c|cc} \lambda & 0 & 0 \\ \hline 0 & 0 & 1 \\ 0 & 0 & 0 \end{array} \right], \ \ 1/2 \le |\lambda| < \sqrt{2}/2$$

Then $w(A) = \rho(A) = |\lambda|$

But $w(A \otimes A) = w(\left[\begin{array}{cc} 0 & 1 \\ 0 & 0 \end{array} \right]) = 1/2 > |\lambda|^2 = w(A)^2$

Prop. $w(A) = \|A\|$ or $w(B) = \|B\| \Rightarrow w(A \otimes B) = w(A)w(B)$

   In parti., A or B normal

Pf.: $\because w(A)w(B) \le w(A \otimes B) \le \|A\|w(B)$

Prop. $w(A \otimes B) = w(A)w(B) \Leftrightarrow A \cong [\lambda] \oplus A'$,

where $|\lambda| = w(A) = \rho(A)$ &

$w(A' \otimes B) \leq |\lambda|w(B)$

or A & B switched

Prop. $w(A \otimes B) = w(A)w(B) \ \forall B \Leftrightarrow A \cong [\lambda] \oplus A'$,

where $|\lambda| = w(A) = \|A'\|$

$$\text{(b) } w(A \otimes B) = \|A\| w(B)$$

Note: $\|A\| \leq 1 \Rightarrow A \cong U \oplus A'$, where U unitary $\&$ A' c.n.u.

Prop.1. A $n \times n$, $\|A\| = 1$, A has unitary part, B $m \times m$

$$\Rightarrow w(A \otimes B) = w(B)$$

$$J_n = \begin{bmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & \ddots & 1 \\ & & & & 0 \end{bmatrix} \quad n \times n \text{ Jordan block}$$

$\Rightarrow W(J_n) = \{z \in \mathbb{C} : |z| \leq \cos(\pi/(n+1))\}$

Prop.2. $W(J_n \otimes J_m) = W(J_l)$, where $l = \min\{n, m\}$

$\qquad w(J_n \otimes J_m) = \min\{w(J_n), w(J_m)\}$

$$\text{Ex. } J_3 \otimes J_2 = \begin{array}{c} \\ \left[\begin{array}{cc|cc|cc} 0 & 0 & 0 & 1 & & \\ 0 & 0 & 0 & 0 & & \\ \hline & & 0 & 0 & 0 & 1 \\ & & 0 & 0 & 0 & 0 \\ \hline & & & & 0 & 0 \\ & & & & 0 & 0 \end{array} \right] \begin{array}{c} \\ \leftarrow \\ \\ \leftarrow \\ \\ \\ \end{array} \end{array}$$

$\Rightarrow J_2$ is submatrix of $J_3 \otimes J_2$

Prop.3. A $n \times n$, $B = \begin{bmatrix} 0 & B_1 & & \\ & 0 & \ddots & \\ & & \ddots & B_k \\ & & & 0 \end{bmatrix}$, $\|A\| = \|A^k\| = 1$

Then $w(A \otimes B) = \|A\| w(B)$

Pf.: Let $x \in \mathbb{C}^n$ be $\ni \|x\| = 1$ & $\|A^k x\| = 1$

Let $y = [y_1 ... y_{k+1}]^T \ni \|y\| = 1$ & $|\langle By, y \rangle| = w(B)$

Let $u = [y_1 \otimes A^k x \ \ y_2 \otimes A^{k-1} x \ ... \ y_{k+1} \otimes x]^T$

Then $\|u\| = 1$ & $|\langle (B \otimes A)u, u \rangle| = |\langle By, y \rangle| = w(B)$

$\Rightarrow w(B) \leq w(B \otimes A) = w(A \otimes B)$

$\Rightarrow$ "="

Thm.1. A $n \times n$, $\|A\| = 1$, B $m \times m$, $w(A \otimes B) = w(B)$

Then either A has unitary part

or A c.n.u. $\&$ $W(B) =$ circular disc at 0

Sketch of Proof.

(1) May assume A c.n.u.

Dilate A to $\underbrace{A' \oplus \cdots \oplus A'}_{\text{rank } (I_n - A^* A)}$, $A' \in S_l$ $(l \leq n)$

(2) May assume $A \in S_n$

Dilate A to infinitely many $(n+1) \times (n+1)$ unitary U_j ∋
$$\sigma(U_i) \cap \sigma(U_j) = \phi \ \ \forall i \neq j$$

(3) Anderson (1970s):

B $m \times m$, $W(B) \subseteq \overline{\mathbb{D}}$ (circular disc) $\&$ $\sharp(\partial W(B) \cap \partial \mathbb{D}) > m$
$\Rightarrow W(B) = \overline{\mathbb{D}}$

$A \; n \times n, \;\; B \; m \times m$

Def. A dilates to B if $B \cong \left[\begin{array}{cc} A & * \\ * & * \end{array} \right]$

(A compression of B; B dilation of A)

Necessarily, $n \leq m$ & $W(A) \subseteq W(B)$

A $n \times n$

Def. $A \in S_n$ if $\|A\| \leq 1$, $\sigma(A) \subseteq \mathbb{D} \equiv \{z \in \mathbb{C} : |z| < 1\}$ &
$$\text{rank } (I_n - A^*A) = 1$$

finite-dim version of $S(\phi)$: Sarason (1967)

Sz.-Nagy–Foias contraction theory (1960s–'70s)

Ex. $A = J_n$
$$A \in S_2 \Leftrightarrow A \cong \left[\begin{array}{cc} a & \sqrt{(1 - |a|^2)(1 - |b|^2)} \\ 0 & b \end{array} \right],$$
where $|a|$, $|b| < 1$

Thm. $A \in S_n$ Then $\forall |\lambda| = 1$, $\exists$ unique $(n + 1) \times (n + 1)$ unitary
$$U \ni$$

(a) A dilates to U,
(b) $\sharp(\partial W(A) \cap \partial W(U)) = n + 1$,
(c) $\lambda \in \sigma(U)$.

Thm.2. A $n \times n$, $\|A\| = 1$. Then the following are equiv.

(a) $W(A \otimes J_m) = W(J_m)$,

(b) $w(A \otimes J_m) = w(J_m)$,

(c) $A \otimes J_m \cong J_m \oplus B$ for some $B \ni w(B) \leq w(J_m)$,

(d) $\|A^{m-1}\| = 1$.

Moreover, for $n = m$, also equiv. to:

(e) either A has unitary part or $A \in S_n$,

(f) $\|A^k\| = 1 \ \forall k \geq 1$ or $\|A^{n-1}\| = 1$ & $\|A^n\| < 1$.

Main parts: (b) $\Rightarrow$ (c) & (d) $\Rightarrow$ (c)

Thm.3. A $n \times n$, $\|A\| = \|A^k\| = 1$ for some $k \geq 1$

Then $w(A) \geq \cos(\pi/(k+2))$

Moreover, the following are equiv.:

(a) $w(A) = \cos(\pi/(k+2))$,

(b) $A \cong J_{k+1} \oplus B$, $w(B) \leq \cos(\pi/(k+2))$,

(c) $W(A) = \{z \in \mathbb{C} : |z| \leq \cos(\pi/(k+2))\}$.

Proof for "$\geq$": $w(A) \geq w(A \otimes J_{k+1}) = w(J_{k+1}) = \cos(\pi/(k+2))$

Main part: (a) $\Rightarrow$ (b).

Special case: $k = 1$ (Williams & Crimmins, 1967)

$$w(A) \geq \|A\|/2$$

Moreover, $w(A) = \|A\|/2 \Leftrightarrow A \cong \|A\|(J_2 \oplus B), \quad w(B) \leq 1/2$

Special case: $k = n - 1$

A $n \times n$, $\|A\| = 1$. Then the following are equiv.:

(a) $\|A^{n-1}\| = 1$ & $w(A) = \cos(\pi/(n+1))$,

(b) $A \cong J_n$,

(c) $\|A^{n-1}\| = 1$ & $A^n = 0$,

(d) $W(A) = \{z \in \mathbb{C} : |z| \leq \cos(\pi/(n+1))\}$ (Wu, 1998)

$$B = [b_{ij}]_{i,j=1}^m$$

Def. $B \succcurlyeq 0$ if $b_{ij} \geq 0 \ \forall i, j$

B (permutationally) reducible if either $B = [0]$ or

B is permutationally similar to $\begin{bmatrix} C & D \\ 0 & E \end{bmatrix}$ (C, E square matrices)

Otherwise, B (permutationally) irreducible

Ex. J_n reducible

$\mathrm{Re} J_n$ ($\equiv (J_n + J_n^*)/2$) irreducible

Thm.4. A $n \times n$, B $m \times m$, $B \succcurlyeq 0$ & ReB irreducible

Then $w(A \otimes B) = \|A\|w(B) \Leftrightarrow$

either $\|A^k\| = \|A\|^k$ $\forall k \geq 1$ or B is permutationally similar

to block-shift matrix $\begin{bmatrix} 0 & B_1 & & \\ & 0 & \ddots & \\ & & \ddots & B_l \\ & & & 0 \end{bmatrix}$ with $\|A^l\| = \|A\|^l$

Tam & Yang (1999):

B $m \times m$, $B \succcurlyeq 0$ & ReB irreducible

Then B permutationally similar to block-shift matrix

$\Leftrightarrow W(B) =$ circular disc at 0.

Note 1. False if $B \succcurlyeq 0$, but ReB reducible

$$\text{Ex. } A = B = \left[\begin{array}{cc|c} 0 & 1 & \\ 0 & 0 & \\ \hline & & a \end{array}\right],\ 0 < a \le 1/2$$

Then $w(A \otimes B) = 1/2 = \|A\| w(B)$

But B not permutationally similar to block-shift

Note 2. False if $B \ngeqslant 0$ with ReB irreducible

$$\text{Ex. } A = J_3,\ B = \left[\begin{array}{ccc} 0 & -\sqrt{2} & 1 \\ 0 & 0 & 1 \\ 0 & 0 & \sqrt{2}/2 \end{array}\right].$$

Then $w(A \otimes B) = \overline{\mathbb{D}} = W(B)$

$\therefore w(A \otimes B) = 1 = \|A\| w(B)$

But B not permutationally similar to block-shift

Note: Any operator A dilates to normal N (Halmos, 1950)

Pf.: $\|A\| \le 1 \Rightarrow \begin{bmatrix} A & (I - AA^*)^{1/2} \\ (I - A^*A)^{1/2} & -A^* \end{bmatrix}$ is unitary

Prop. A dilates to normal $N \ni \sigma(N) \subseteq \sigma(A)$ Then

    (a) $\|A\| = \|N\| = \rho(N) = \rho(A) = w(N) = w(A)$,

    (b) $\overline{W(A)} = \overline{W(N)} = \sigma(N)\hat{} = \sigma(A)\hat{}$,

    (c) $\overline{W(A \otimes B)} = (\overline{W(A)W(B)})\hat{} \;\; \forall B$,

    (d) $w(A \otimes B) = w(A)w(B) = \|A\|w(B) \;\; \forall B$

Ex. A subnormal (extended to normal N: $N \cong \begin{bmatrix} A & * \\ 0 & * \end{bmatrix}$),

    A Toeplitz ($Af = P_{H^2}(\phi f)|H^2$ on H^2: $\phi \in L^\infty(\partial\mathbb{D})$),

    $W(A) =$ triangular region (Mirman, 1968)

Def. A hyponormal if $A^*A \geq AA^*$

$\therefore A$ hyponormal $\Leftarrow A$ subnormal $\Leftarrow A$ normal

Thm. A hyponormal $\Rightarrow A$ dilates to normal N

$$\ni \sigma(N) \subseteq \sigma_l(A) \cap \sigma_r(A) \subseteq \sigma(A)$$

Pf.: Use Sz.-Nagy–Foias theory

Cor. A hyponormal $\Rightarrow w(A \otimes B) = \|A\|w(B) = w(A)w(B) \; \forall B$

Note 1. Confirms Shiu's conjecture (1978)

Note 2. A hyponormal, Toeplitz or $W(A) =$ triangular region

$\Rightarrow\!\!\!\!/ \; w(A \otimes B) = w(A)\|B\| \; \forall B$

Ex. A as above

$B = J_n$

Then $w(A \otimes B) = w(A)w(B) < w(A)\|B\|$

$\qquad\qquad\qquad\qquad\quad \| \qquad\qquad \|$

$\qquad\qquad\qquad \cos(\pi/(n+1)) \qquad 1$

References:

[1] H.-L. Gau, K.-Z. Wang & P.Y.Wu, Numerical radii for tensor products of matrices, perprint.

[2] H.-L. Gau, K.-Z. Wang & P.Y.Wu, Numerical radii for tensor products of operators, perprint.

「矩陣與算子國際研討會」、「矩陣理論和環理論國際研討會」報告

國立交通大學應用數學系

吳培元

本人此次赴大陸參加兩項接續舉辦的國際研討會:2013 年 7 月 15 日至 18 日在北京市「北京師範大學」的「矩陣與算子國際研討會」(International Workshop on Matrices and Operators :MAO 2013)，和 7 月 19 日至 22 日在南京市「東南大學」的「矩陣理論和環理論國際研討會」(The International Workshop on Matrix Theory and Ring Theory)。

我是在 7 月 15 日下午搭乘長榮航空班機自「桃園機場」來到北京市的「首都國際機場」，當日即由「北京師範大學」的兩位研究生接機後，進駐該校的「京師大廈」。正式會議在第二天，7 月 16 日早上九點開始進行。地點在該校的圖書館主樓的十一樓教室。八點半先舉行了開幕式，大家步行到樓下合照全體照。會議演講分兩組平行進行，分別是中文組和英文組。我參加的是後一組，上午有四場演講，每人三十分鐘，分別是 T.Ando, R.A. Brualdi, M.-D. Choi 和 N.-C. Wong。其中 Ando 已經八十幾歲了，是這一次與會者中最年長的一位。他的講題是〝Majorization relations involving partial traces〞，和其他的演講內容有部分關連。下午兩點起，有五場演講，分別是譚天佑、Yongdo

Lim、Ajda Fosner、高華隆和張其棟，高和張兩人報告的都是數值域方面的題材，前者的是和我合作的論文結果。

第二天，7月17日，演講仍是以中英文兩組平行進行。我在英文組九點到九點半，給第一個演講〝Numerical ranges of KMS matrices〞，這是根據一篇已被〝Acta Sci. Math. (Szeged)〞接受的論文內容作的演講。演講後，座下的李志光給了不少改進的意見，非常感激他。上午的其他三位演講者是譚必信、蘇華先和潘耀東，後兩位是我主持的。下午的五位演講者是施能聖、簡茂丁、詹興致、李志光和李中山，其中詹的演講是最弱的，三十分鐘的演講只講了二十分鐘。兩天的會議議程於下午五點結束。

7月18日是旅遊時間，由「北京師範大學」招待參觀「八達嶺長城」和「明十三陵」中的「定陵」。

我在7月19日離開「北京市」，搭機到「南京市」的「祿口國際機場」。由「東南大學」的兩位研究生接機，搭乘計程車。到該校的「榴園賓館」進駐，準備次日的下一場會議。

「矩陣理論和環理論國際研討會」在7月20日上午開始，進行兩天的議程，這是第一次遇到將「矩陣理論」和「環理論」兩者放在一起的會議，與會者在這兩個領域各佔一半。外賓中有羅馬尼亞的 T. Albu，捷克的 J. Trlifaj，和印度的 R.B. Bapat。後者作的是矩陣相關的

議題。我的演講〝Explorations in numerical ranges〞被安排在下午四點到四點半，我介紹了我和高華隆近年所作的關於上一世紀六零到七零年代的三個古典題材 (J.Anderson, J.Holbrook, 和 J.P. Williams & T. Crimmins) 的數值域相關結果，反應還不錯。

第二天，7 月 21 日的演講分成「環理論」和「矩陣理論」兩組平行進行，我也主持了上午的魏益民、許慶祥等人的三個演講，每人三十分鐘，整個會議議程到下午五點半才結束。其後兩組合併舉行了一場會議結業式，由與會的外賓分別感謝主辦單位的辛勞。

7 月 22 日是旅遊時間。由主辦當局安排遊覽車作「南京市」的旅遊，我參加了上午前往「中山陵」和「靈谷寺」的行程。只可惜因當天是星期一，前者沒有開放參觀，後者也因時間緊迫而沒有入內。因為我要搭乘今晚的班機回台，故下午到「總統府」和「玄武湖」的行程就沒有辦法參加，改由該校數學系系主任陳建龍陪我在「東南大學」校園內參觀。該校是台灣「中央大學」的前身，最有名的校友是吳健雄女士，故校園內有「吳健雄紀念館」、「吳健雄實驗室」和「健雄院」等。

當天傍晚時分，我搭車到「祿口國際機場」搭程「長榮航空」的班機，回到台灣，結束了前後八天(7 月 15 至 22 日)的「北京」和「南京」之旅。回到「新竹市」家中，都已是晚上十一點半了。

Numerical Ranges of KMS Matrices

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Beijing Normal University

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I. Numerical Range

A $n \times n$ complex matrix

Numerical range of A:

$$W(A) = \{\langle Ax, x \rangle : x \in \mathbb{C}^n, \|x\| = 1\}$$

Numerical radius of A:

$$w(A) = \max\{|z| : z \in W(A)\}$$

$$x = [x_1, ..., x_n]^T, \ y = [y_1, ..., y_n]^T$$

$$\Rightarrow \langle x, y \rangle = x_1 \overline{y_1} + \cdots + x_n \overline{y_n}$$

$$\|x\| = (|x_1|^2 + \cdots + |x_n|^2)^{1/2}$$

Properties:

(1) $W(A)$ nonempty compact subset of $\mathbb{C}$

(2) U $n \times n$ unitary $\Rightarrow W(U^*AU) = W(A)$

(3) Toeplitz-Hausdorff (1918–19):
$W(A)$ convex

(4) $\sigma(A) \subseteq W(A)$

(5) $\|A\|/2 \leq w(A) \leq \|A\|$ $(\equiv \max\{\|Ax\| : x \in \mathbb{C}^n, \|x\| = 1\})$

II. KMS Matrix

(Upper-triangular) KMS matrix:

$$J_n(a) = \begin{bmatrix} 0 & a & a^2 & \cdots & a^{n-1} \\ & 0 & a & \ddots & \vdots \\ & & \ddots & \ddots & a^2 \\ & & & \ddots & a \\ & & & & 0 \end{bmatrix} \quad (n \geq 1, a \in \mathbb{C})$$

Kac, Murdock & Szegö (1953):

Poisson kernel:

$$P_a(\theta) = \text{Re}\left(\frac{1+ae^{i\theta}}{1-ae^{i\theta}}\right) = \frac{1-a^2}{1-2a\cos\theta+a^2}$$

$$= \sum_{k=-\infty}^{\infty} a^{|k|} e^{ik\theta} \quad (0 \leq a < 1, 0 \leq \theta < 2\pi)$$

Harmonic extension of $f \in L^1(\partial\mathbb{D})$

$$\tilde{f}(ae^{i\theta}) = \frac{1}{2\pi} \int_0^{2\pi} P_a(\theta - t) f(e^{it}) dt \quad \text{on } \mathbb{D} \equiv \{z \in \mathbb{C} : |z| < 1\}$$

nth Toeplitz matrix:

$$\begin{bmatrix} 1 & a & \cdots & a^{n-1} \\ a & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & a \\ a^{n-1} & \cdots & a & 1 \end{bmatrix} = 2 \operatorname{Re} J_n(a) + I_n$$

General question:

What can we say about $W(J_n(a))$?

Note: $J_n(a)$: meeting ground of

$\left\{\begin{array}{l} \text{nilpotent matrices,} \\ \text{Toeplitz matrices,} \\ \text{nonnegative matrices,} \\ S_n\text{-matrices,} \\ S_n^{-1}\text{-matrices.} \end{array}\right.$

Lma. (1) $|a| = |b| \Rightarrow J_n(a) \cong J_n(b)$ (unitarily similar)

(2) $a, b \neq 0 \Rightarrow J_n(a) \approx J_n(b)$ (similar)

(3) $W(J_n(a))$ symmetric w.r.t. $x-$axis

(4) $n \geq 2, a \neq 0 \Rightarrow J_n(a)$ irreducible $(J_n(a) \not\cong A \oplus B)$,
$$0 \in \text{Int } W(J_n(a)),$$
$$\partial W(J_n(a)) \text{ differentiable}$$

(5) $|a| \leq |b| \Rightarrow W(J_n(a)) \subseteq W(J_n(b))$

Note: $(1) \Rightarrow$ may assume $a \geq 0$

III. Circular Disc

Thm. $W(J_n(a))$ is circular disc (centered at 0) $\Leftrightarrow n = 2$ & $a \neq 0$

Pf.: "$\Leftarrow$":

$$J_2(a) = \left[\begin{array}{cc} 0 & a \\ 0 & 0 \end{array} \right]$$

$$\Rightarrow W(J_2(a)) = \{z \in \mathbb{C} : |z| \leq |a|/2\}$$

"$\Rightarrow$":
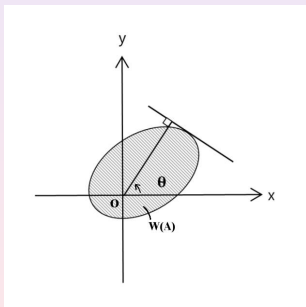
Kippenhahn polynomial of $n \times n$ A

$$p_A(x, y, z) = \det \left(x\text{Re } A + y\text{Im } A + zI_n \right)$$

Then $p_A(1, i, -z) = \det (A - zI_n)$

zeros are eigenvalues of A

$p_A(\cos\theta, \sin\theta, -z) = \det (\text{Re}(e^{-i\theta}A) - zI_n)$

max zero is distance from 0 to supporting line of $W(A)$

$"\Rightarrow"$ Assume $n \geq 3$

Let $W(J_n(a)) = \{z \in \mathbb{C} : |z| \leq r\}$

Bézout $\Rightarrow q(x, y, z) \equiv z^2 - r^2(x^2 + y^2)$ divides $p_A(x, y, z)$

Let $r(x, y, z) = \frac{p_A(x,y,z)}{q(x,y,z)}$

$\therefore r(1, y, 0) = \frac{p_A(1,y,0)}{q(1,y,0)} = \frac{\det(\text{Re}A + y\text{Im}A)}{-r^2(1+y^2)}$

computation $\Rightarrow = (-1)^n \frac{1}{4r^2} a^{2n-2}$

But $r(1, i, z) = \frac{p_A(1,i,z)}{q(1,i,z)} = \frac{z^n}{z^2} = z^{n-2}$

$\Rightarrow r(1, i, 0) = 0 \ (\because n \geq 3 \ )$

$\Rightarrow a = 0$

$\Rightarrow W(J_n(a)) = \{0\}$ not circular disc $\rightarrow\leftarrow$

IV. S_n- and S_n^{-1}-matrix

Def. A $n \times n$ of class S_n if

(1) $\|A\| \leq 1$,
(2) $\sigma(A) \subseteq \mathbb{D}$, and
(3) rank $(I_n - A^*A) = 1$.

Finite-dim version of $S(\phi)$: Sarason (1967)
Sz.-Nagy–Foias contraction theory (1960s–'70s)

Def. A $n \times n$ of class S_n^{-1} if

(1) $\sigma(A) \subseteq \mathbb{C} \setminus \mathbb{D}$, and
(2) rank $(I_n - A^*A) = 1$.

Gau (2010)

Prop. (a) $0 < |a| < 1 \Rightarrow ((1 - |a|^2)/a) \, J_n(a) - \bar{a} I_n$ is of class S_n

(b) $\qquad |a| > 1 \Rightarrow ((1 - |a|^2)/a) \, J_n(a) - \bar{a} I_n$ is of class S_n^{-1}

Cor. $A \in S_n$ or S_n^{-1}, $\sigma(A) = \{\lambda\}$

Then the following are equiv.:

(a) $W(A)$ is circular disc,

(b) $\partial W(A)$ contains an elliptic arc, and

(c) $n = 2$ or $\lambda = 0$.

V. Boundary Line Segment

Thm. $\partial W(J_n(a))$ contains line segment $\Leftrightarrow n \geq 3$ & $|a| = 1$.
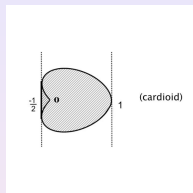
Pf.: Via S_n- & S_n^{-1}-matrices

Ex.1.

$$A \equiv J_n(1) = \begin{bmatrix} 0 & 1 & \cdots & 1 \\ & 0 & \ddots & \vdots \\ & & \ddots & 1 \\ & & & 0 \end{bmatrix}$$
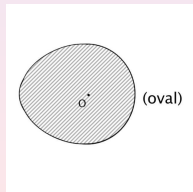
(1) $W(A)$ symm. w.r.t. x-axis

(2) $(-1/2)\, I_n \leq \mathrm{Re}A \leq ((n-1)/2)\, I_n$

(3) $\partial W(A)$ has exactly one line segment on $x = -1/2$

(4) dist$(0, \partial W(A)) = 1/2$ & attained at exactly one pt $(-1/2, 0)$

Ex2.

$$W(J_3(a)) =$$

if $|a| = 1$,

if $|a| \neq 0, 1$.

VI. Compression

$A \ m \times m$, $B \ n \times n$

Def. A is compression of B (or B is a dilation of A) if

$$B \cong \left[\begin{array}{cc} A & * \\ * & * \end{array} \right]$$

Necessaily, $m \le n$ & $W(A) \subseteq W(B)$

Thm.1. $1 \le m < n$, $a \ne 0$, $A \ m \times m$ compression of $J_n(a)$

$\Rightarrow W(A) \subsetneqq W(J_n(a))$

Pf.: Via S_n- & S_n^{-1}-matrices

Thm.2. $n \geq 2, |a| \neq 0, 1, J_n(a) \cong \begin{bmatrix} A & * \\ 0 & * \end{bmatrix}$, $A \ m \times m \ (1 \leq m < n)$

$\Rightarrow W(A) \subseteq \text{Int } W(J_n(a))$

Cor. $1 \leq m < n, |a| \neq 0, 1 \Rightarrow W(J_m(a)) \subseteq \text{Int } W(J_n(a))$

Lma. $a \neq 0, x \in \mathbb{C}^n, \|x\| = 1 \ni \langle J_n(a)x, x \rangle \in \partial W(J_n(a))$, not on line segment

$\Rightarrow x$ cyclic vector for $J_n(a)$.

Def. x cyclic for $n \times n \ A$ if $\bigvee \{x, Ax, ..., A^{n-1}x\} = \mathbb{C}^n$

Pf. of Lma.:

$|a| \neq 1$: via S_n- & S_n^{-1}-matrices

$|a| = 1$: separate arguments

Thm.3. $n \geq 2, |a| < |b| \Rightarrow W(J_n(a)) \subseteq \text{Int } W(J_n(b))$

VII. Principal Submatrix

A $n \times n$, $1 \leq j \leq n$

$A[j] = (n-1) \times (n-1)$ submatrix of A by deleting its jth row
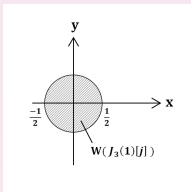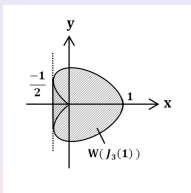& jth column

Thm. $n \geq 2, |a| \neq 0, 1, b = \min \sigma(\text{Re } J_n(a))$, $1 \leq j \leq n$

$\Rightarrow \partial W(J_n(a)) \cap \partial W(J_n(a)[j]) = \{b\}$ if n is odd, $j = (n+1)/2$

& $|a| > 1$; otherwise, $W(J_n(a)[j]) \subseteq \text{Int } W(J_n(a))$

Note: (1) $|a| = 1 \Rightarrow \partial W(J_3(a)) \cap \partial W(J_3(a)[j]) = \{-1/2\}$ $\forall j$

(2) $n \geq 4, |a| = 1 \Rightarrow \partial W(J_n(a)) \cap \partial W(J_n(a)[j])$

$= $ line segment on $\partial W(J_{n-1}(a)) \forall j$

Ex. $J_3(1) = \begin{bmatrix} 0 & 1 & 1 \\ & 0 & 1 \\ & & 0 \end{bmatrix}$, $\quad J_3(1)[j] = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad \forall j$

More generally,

Thm. A S_n-matrix $n \geq 2$

(1) B $m \times m$ $(1 \leq m < n)$ compression of $A \Rightarrow W(B) \subsetneqq W(A)$

(2) $A \cong A' \equiv [a_{ij}]_{i,j=1}^n$, $|a_{ii}| < 1$ $\forall i$,

$$a_{ij} = \begin{cases} (1 - |a_{ii}^2|)^{1/2}(1 - |a_{jj}|^2)^{1/2} \prod_{k=i+1}^{j-1}(-\bar{a}_{kk}) & \text{if } i < j, \\ 0 & \text{if } i > j. \end{cases}$$

$\Rightarrow W(A'[j]) \subseteq \text{Int } W(A')$ $\forall j$

Lma. A $n \times n$

(1) dim $\bigvee\{x \in \mathbb{C}^n : \langle Ax, x \rangle = a\|x\|^2\} = 1$ $\forall a \in \partial W(A)$

   & $\bigvee\{x \in \mathbb{C}^n : \langle Ax, x \rangle = a\|x\|^2, a \in \partial W(A)\} = \mathbb{C}^n$

   B $m \times m$ $(1 \le m < n)$ compression of A

   $\Rightarrow W(B) \subsetneqq W(A)$

(2) $x = [x_1, ..., x_n]^T \in \mathbb{C}^n, \|x\| = 1, \langle Ax, x \rangle \in \partial W(A) \Rightarrow x_j \ne 0$ $\forall j$

   Then $W(A[j]) \subseteq \text{Int } W(A)$ $\forall j$

Note 1. $A \in S_n^{-1}, B \ m \times m \ (1 \le m < n)$ compression of A
$$\Rightarrow W(B) \subsetneq W(A)$$

Note 2. Thm.1 true for $A \in S_n^{-1}$ with $\sigma(A)$ a singleton
False for general S_n^{-1}-matrices

Ex. $A = \begin{bmatrix} 2 & 2\sqrt{3} & 6-12i \\ 0 & 1+2i & 4\sqrt{3} \\ 0 & 0 & 2-3i \end{bmatrix} \in S_3^{-1}$ in standard form

$W(A[j]) \subseteq \text{Int } W(A) \ \forall j$

VIII. Zero-dilation Index

A $n \times n$

zero-dilation index of A:

$$d(A) = \max\left\{k \geq 1 : A \cong \begin{bmatrix} 0_k & * \\ * & * \end{bmatrix}\right\}$$

Gau. Wang & Wu (?)

A normal or weighted permutation matrix with zero diagonals

rank-k numerical range of A

$$\Lambda_k(A) = \left\{\lambda \in \mathbb{C} : A \cong \begin{bmatrix} \lambda I_k & * \\ * & * \end{bmatrix}\right\} \quad (1 \leq k \leq n)$$

Li-Sze characterization (2008):

$$\Lambda_k(A) = \bigcap_{\theta \in \mathbb{R}} \left\{\lambda \in \mathbb{C} : \text{Re}(e^{-i\theta}\lambda) \leq \lambda_k \left(\text{Re } (e^{-i\theta}A)\right)\right\}$$

In parti., $d(A) = \min\{i_{\geq 0}(\text{Re}(e^{-i\theta}A)): \theta \in \mathbb{R}\}$

B $n \times n$ Hermitian matrix:

$\lambda_1(B) \geq \cdots \geq \lambda_n(B)$: eigenvalues of B

$i_{\geq 0}(B) =$ no. of nonnegative eigenvalues of B

Thm. $n \geq 2, a \in \mathbb{C}$

$\Rightarrow d(J_n(a)) = i_{\geq 0}(\text{Re} J_n(a))$

$$= \begin{cases} n & \text{if } a = 0, \\ k & \text{if } \cos(k\pi/(n-1)) < |a| \leq \cos((k-1)\pi/(n-1)) \\ & (1 \leq k \leq \lfloor n/2 \rfloor), \\ 1 & \text{if } |a| > 1. \end{cases}$$

A, B $n \times n$ Hermitian

Def. A & B are congruent if $\exists$ invertible $X \ni XAX^* = B$

Sylvester's law of inertia:

    A, B congruent $\Rightarrow d(A) = d(B)$

Lma. 1. $n \geq 2$ & $a \neq 0 \Rightarrow$ Re $(e^{-i\theta} J_n(a))$ congruent to $H_n(a, \theta)$
                           $\forall \theta \in \mathbb{R}$

$$H_n(a, \theta) = \begin{bmatrix} -2|a|\cos\theta & 1 & & \\ 1 & \ddots & \ddots & \\ & \ddots & -2|a|\cos\theta & 1 \\ & & 1 & 0 \end{bmatrix}$$

Lma. 2. $n \geq 3$, $a \in \mathbb{C}$, $\theta \in \mathbb{R}$

(1) $0 \in \sigma(H_n(a, \theta)) \Leftrightarrow |a| \cos \theta = \cos((j\pi)/(n-1))$, $1 \leq j \leq n-2$.

(2) $i_{\geq 0}(H_n(a, \theta)) = i_{\geq 0}((\text{Re} J_{n-2}) - (|a| \cos \theta) I_{n-2}) + 1$

(3) $i_{\geq 0}(H_n(a, \theta_1)) \leq i_{\geq 0}(H_n(a, \theta_2))$, $0 \leq \theta_1 \leq \theta_2 \leq \pi$.

Pf. of Thm:

$$d(J_n(a)) = i_{\geq 0}(H_n(a, 0)) = i_{\geq 0}(\text{Re} J_n(a))$$
$$\|$$
$$i_{\geq 0}((\text{Re} J_{n-2}) - (|a| \cos \theta) I_{n-2})$$
$$\|$$
$$(k-1) + 1$$
$$\|$$
$$k$$

References:

KMS matrices

[1] H.-L. Gau & P.Y. Wu, Numerical ranges of KMS matrices*,
    Acta Sci. Math. (Szeged), to appear.
    (*Dedicated to the memory of Prof. B. Sz.-Nagy on his 100th
      birthday)

Zero-dilation indices:

[2] H.-L. Gau, K.-Z. Wang & P.Y. Wu, Zero-dilation index of a
    finite matrix, arXiv:1304.0296.

[3] H.-L. Gau & P.Y. Wu, Zero-dilation indices of KMS matrices*,
    arXiv:1304.0295.
    (* Dedicated to Prof. T. Ando with admiration)

Compressions:

[4] H.-L. Gau & P.Y. Wu, Numerical ranges and compressions of
    S_n-matrices, Oper. Matrices, to appear.

Explorations in Numerical Ranges

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A $n \times n$ complex matrix

(1) $W(A) = \{\langle Ax, x \rangle : x \in \mathbb{C}^n, \|x\| = 1\}$: numerical range of A

$$x = [x_1 ... x_n]^T, \ y = [y_1 ... y_n]^T$$

$$\Rightarrow \langle x, y \rangle = x_1 \overline{y_1} + \cdots + x_n \overline{y_n}$$

$$\|x\| = (|x_1|^2 + \cdots + |x_n|^2)^{1/2}$$

(2) $w(A) = \max\{|z| : z \in W(A)\}$: numerical radius of A

Properties:

(1) $W(A)$ nonempty compact subset of $\mathbb{C}$

(2) U unitary $\Rightarrow W(U^*AU) = W(A)$

(3) $A \cong \begin{bmatrix} B & * \\ * & * \end{bmatrix} \Rightarrow W(B) \subseteq W(A)$

(4) $\lambda \in W(A) \Leftrightarrow A \cong \begin{bmatrix} \lambda & * \\ * & * \end{bmatrix}$

(5) Toeplitz–Hausdorff (1918–19):
   $W(A)$ convex

(6) $\sigma(A) \subseteq W(A)$

(7) $W(A \oplus B) = (W(A) \cup W(B))\check{\ }$

(8) A normal $\Rightarrow W(A) = \sigma(A)\check{\ }$

(9) $\|A\|/2 \le w(A) \le \|A\|$ $(\equiv \max\{\|Ax\| : x \in \mathbb{C}^n, \|x\| = 1\})$

Question 1.   Given A, what can we say about $W(A)$?

Question 2.   Known $W(A)$, what can we say about A ?

Question 3.   Which bdd convex $\triangle \subseteq \mathbb{C}$ is $W(A)$ for some A ?

Ex.1.

$$A = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$$

$\Rightarrow W(A) =$ elliptic disc with foci a, c & minor axis of
length $|b|$

Ex.2.

$$A = \begin{bmatrix} a_1 & & 0 \\ & \ddots & \\ 0 & & a_n \end{bmatrix}$$

$\Rightarrow W(A) = $ polygonal region with some of the a_j's

as vertices

Ex.3.

$$A = J_n \equiv \begin{bmatrix} 0 & 1 & & \\ & 0 & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{bmatrix} \quad (n \times n \text{ Jordan block})$$

$\Rightarrow W(A) = $ circular disc centered at 0 & radius
$$\cos(\pi/(n+1))$$

(I) J. Anderson (early 1970s):

Condition for $W(A)$ = circular disc

(II) J. Holbrook (1969):

Inequality of $w(AB)$ for $AB = BA$

(III) J. P. Williams & T. Crimmins (1967):

Condition for $w(A) = \|A\|/2$

$$\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$$

(I) J. Anderson (early 1970s):

Thm. A $n \times n$, $W(A) \subseteq \overline{\mathbb{D}}$ & $\#\,(\partial W(A) \cap \partial \mathbb{D}) > n \Rightarrow W(A) = \overline{\mathbb{D}}$

Pf: (Wu, 1999)

$W(A) \subseteq \overline{\mathbb{D}} \Leftrightarrow Re\,(e^{-i\theta} A) \leq I_n \quad \forall\,\theta \in \mathbb{R}$

Let $p(e^{i\theta}) = \det\,(I_n - Re\,(e^{-i\theta} A)) \quad \forall\,\theta$

Then $p(e^{i\theta}) = \bar{a}_n e^{-in\theta} + \cdots + \bar{a}_1 e^{-i\theta} + a_0 + a_1 e^{i\theta} + \cdots + a_n e^{in\theta}$
$\geq 0 \quad \forall\theta$

Moreover, $e^{i\theta} \in \partial W(A) \Leftrightarrow p(e^{i\theta}) = 0$

$\therefore p(e^{i\theta}) = |q(e^{i\theta})|^2$ for some poly. q of deg.$\leq n$ (Riesz-Fejér, 1916)

$\#\,(\partial W(A) \cap \partial \mathbb{D}) > n \;\Rightarrow p(e^{i\theta}) = 0$ for more than n θ's
$\Rightarrow q(e^{i\theta}) = 0$ for more than n θ's

$\Rightarrow q \equiv 0 \qquad \Rightarrow p \equiv 0 \qquad \Leftrightarrow W(A) = \overline{\mathbb{D}}$

Cor. $\{z \in \overline{\mathbb{D}} : Re\,z \geq 0\} \neq W(A) \qquad \forall$ finite matrix A

Generalizations:

(1) Gau & Wu (2004):

$$A = \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \\ -a_n & \cdots & -a_2 & -a_1 \end{bmatrix} \quad n \times n \text{ companion matrix}$$

$W(A) \supseteq D$ (closed circular disc at 0),
$\#(\partial W(A) \cap \partial D) > n \Rightarrow A = J_n$

(2) Gau & Wu (2006):

A compact, $W(A) \subseteq \overline{\mathbb{D}}$, $\#(\partial W(A) \cap \partial \mathbb{D}) = \infty \Rightarrow W(A) = \overline{\mathbb{D}}$
Idea: "analytic" branch of $d_A(\theta) = \max W(Re\ (e^{-i\theta}A))$, $\theta \in \mathbb{R}$
   $\because d_A(\theta) \leq 1 \quad \forall\ \theta$
   $d_A(\theta) = 1 \quad$ for infinitely many θ's $\qquad \Rightarrow d_A \equiv 1$

Cor. $\{z \in \overline{\mathbb{D}} : Re\ z \geq 0\} \neq W(A)$ for A compact

(3) Gau & Wu (2008):

A $n \times n$, $W(A) \supseteq \overline{\mathbb{D}}$, $\#(\partial W(A) \cap \partial \mathbb{D}) > n$

$\Rightarrow \partial W(A)$ contains an arc of $\partial \mathbb{D}$

Ex. $A = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} \oplus \text{diag}\ (r, re^{i\theta_0}, \ldots, re^{i(n-3)\theta_0})$

$1 < r < \sec\ (\pi/(n-2))$, $\theta_0 = 2\pi/(n-2)$

$\Rightarrow W(A) \supseteq \overline{\mathbb{D}}$ & $\partial W(A)$ contains $n-2$ arcs of $\partial \mathbb{D}$

(4) A $n \times n$ nilpotent, $W(A) \subseteq \overline{\mathbb{D}}$, $\#(\partial W(A) \cap \partial \mathbb{D}) > n-2$

$\Rightarrow W(A) = \overline{\mathbb{D}}$ & "$n-2$" is sharp:

Ex. $A_n = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 1 \\ & 0 & 1 & & & 0 \\ & & \ddots & \ddots & & \vdots \\ & & & \ddots & \ddots & 0 \\ & & & & \ddots & 1 \\ & & & & & 0 \end{bmatrix}$ $n \times n$ $(n \geq 3)$

$\Rightarrow W(A_n) \subsetneqq \overline{\mathbb{D}}$ & $\#(\partial W(A_n) \cap \partial \mathbb{D}) = n-2$

(5) A $n \times n$ nilpotent, $W(A) \supseteq \overline{\mathbb{D}}$, $\#(\partial W(A) \cap \partial \mathbb{D}) > n-2$

$\Rightarrow \begin{cases} W(A) = \overline{\mathbb{D}} & \text{if } 2 \leq n \leq 4 \\ \partial W(A) \text{ contains an arc of } \partial \mathbb{D} & \text{if } n \geq 5 \end{cases}$

(6) Wu (2011):

A $n \times n$, $W(A) = $ circular disc centered at a

$\Rightarrow a$ is eigenvalue of A with $1 \leq$ geom. multi. $<$ alg. multi.

Cor.1. A $n \times n$ similar to normal $\Rightarrow$ $W(A) \neq$ circular disc

Cor.2. A $n \times n$ nonnegative $\&$ irreducible

$\Rightarrow W(A) \neq$ circular disc

Cor.3. A $n \times n$ row stochastic $\Rightarrow$ $W(A) \neq$ circular disc

Cheung $\&$ Li (2013)

(7) A $n \times n$ $A \cong A_1 \oplus + \cdots \oplus A_k$, $W(A) =$ elliptic disc

   $\Rightarrow W(A_j) =$ elliptic disc for some j

Pf.: $\because W(A) = (W(A_1) \cup \cdots \cup W(A_k))\hat{} =$ elliptic disc E

   pigeonhole principle $\Rightarrow \#(\partial W(A_j) \cap \partial E) = \infty$ for some j

   Anderson $\Rightarrow W(A_j) = E$

Note. Generalize Tam $\&$ Yang (1999)

(II) J. Holbrook (1969):

$$AB = BA \overset{?}{\Rightarrow} w(AB) \leq w(A)\|B\|, \ \|A\|w(B)$$

Known:

(1) $w(AB) \leq 4w(A)w(B)$ & "4" is sharp:

Ex. $A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \ B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \ \Rightarrow AB = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

$\therefore w(AB) = 1, \ w(A) = w(B) = 1/2$

(2) $AB = BA \Rightarrow w(AB) \leq 2w(A)w(B)$ & "2" is sharp:

Ex. $A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \oplus \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \ B = \begin{bmatrix} 0 & & & \\ 0 & 0 & & \\ 1 & 0 & 0 & \\ 0 & 1 & 0 & 0 \end{bmatrix}$

$w(AB) = w(A) = w(B) = 1/2$

(3) True if A normal (Holbrook, 1969)

if $AB = BA$ & $AB^* = B^*A$ (Holbrook, 1969)

if A, B 2×2 (Holbrook, 1992)

(4) Crabb (1976):

$AB = BA \Rightarrow w(AB) \leq (\sqrt{2 + 2\sqrt{3}}/2) \ w(A)\|B\|$

(5) Müller, Davidson & Holbrook (1988):

Ex. $A = J_9$, $B = J_9^3 + J_9^7$

$w(AB) = \|A\| = 1$, $\|B\| \geq \sqrt{2}$

$w(A) = w(B) = \cos(\pi/10)$

$\Rightarrow w(AB) > \|A\|w(B)$, but $w(AB) \leq w(A)\|B\|$

Holbrook & Schoch (2010): A, B 3×3

Wu, Gau & Tsai (2009) :

Thm 1. $A \in S_n$, $AB = BA \Rightarrow w(AB) \leq w(A)\|B\|$
          But $A \in S_n$, $AB = BA \nRightarrow w(AB) \leq \|A\|w(B)$

Def. $A \in S_n$ if $\|A\| \leq 1$, $\sigma(A) \subseteq \mathbb{D}$ & rank $(I_n - A^*A) = 1$

Ex.1. $J_n \in S_n$

Ex.2. A 2×2. Then $A \in S_2 \Leftrightarrow A \cong \begin{bmatrix} a & \sqrt{(1 - |a|^2)(1 - |b|^2)} \\ 0 & b \end{bmatrix}$,

$\qquad\qquad\qquad\qquad$ where $|a|$, $|b| < 1$

$\qquad$ finite-dim version of $S(\phi)$: Sarason (1967)

$\qquad$ Sz.-Nagy–Foias contraction theory (1960s–'70s)

Cor. A quadratic, $AB = BA \Rightarrow w(AB) \leq w(A)\|B\|$

$$\uparrow$$

$$A^2 + aA + bI = 0 \quad \text{for some} \quad a, b \in \mathbb{C}$$

Unknown:

$$A \text{ quadratic}, \ AB = BA \overset{?}{\Rightarrow} w(AB) \leq \|A\|w(B)$$

Known:

(1) Rao (1994):

$$A^2 = aI, \ AB = BA \Rightarrow w(AB) \leq \|A\|w(B)$$

(2) Gau, Huang & Wu (2008):

$$A^2 = 0 \quad \text{or} \quad A^2 = A, \ AB = BA$$

$$\Rightarrow w(AB) \leq \min\{w(A)\|B\|, \ \|A\|w(B)\}$$

$$\|A\|/2 \leq w(A) \leq \|A\|$$

(III) Williams & Crimmins (1967):

$$\|A\| = 2, \ w(A) = 1$$

$$\Rightarrow A \cong \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} \oplus A' \ \& \ W(A) = \overline{\mathbb{D}}$$

Pf.: Let $x \in \mathbb{C}^n \ni \|x\| = 1 \ \& \ \|Ax\| = \|A\|$

Let $y = (1/2)Ax$

Then $\|y\| = 1 \ \& \ x \perp y$

Let $K = \vee \{y, x\}$

Then $A \cong \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} \oplus A'$ on $H = K \bigoplus K^{\perp}$

August, 1978
Helsinki, Finland
James P. Williams (1938–1983)

Crabb (1971):

$w(A) \leq 1$, $\|A^n x\| = 2$ for some $n \geq 1$ & $\|x\| = 1$

$$\Rightarrow A \cong \begin{cases} \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} \oplus A^{'} & \text{if } n = 1 \\ \begin{bmatrix} 0 & \sqrt{2} & & & & & \\ & 0 & 1 & & & & \\ & & \ddots & \ddots & & & \\ & & & \ddots & 1 & & \\ & & & & \ddots & \sqrt{2} & \\ & & & & & 0 \end{bmatrix} \oplus A^{'} & \text{if } n \geq 2 \end{cases}$$

& $W(A) = \overline{\mathbb{D}}$

Gau & Wu (2009): confirming a conjecture of Drury (2008)

Thm. $f(z) = z \cdot \Pi_{j=1}^{n-1}(z - a_j)/(1 - \overline{a_j}z)$, $|a_j| < 1$ $\forall j$

$w(A) \leq 1$

$\|f(A)\| = 2$

$\Rightarrow A \cong B \oplus A'$, where B similar to the S_{n+1}-matrix with

eigenvalues $a_1, ..., a_{n-1}, 0, 0$ & $W(A) = \overline{\mathbb{D}}$

Special cases:

$f(z) = z \Rightarrow$ Williams & Crimmins

$f(z) = z^n$ $(n \geq 2) \Rightarrow$ Crabb

Gau, Wang & Wu (201?)

A $n \times n$, $\|A\| = \|A^k\| = 1$ for some $k \geq 1$

$\quad \Rightarrow w(A) \geq \cos(\pi/(k+2))$

Moreover, $"=" \Leftrightarrow A \cong J_{k+1} \oplus A'$, $w(A') \leq \cos(\pi/(k+2))$

Special cases:

$k = 1 \Rightarrow$ Williams & Crimmins

$k = n - 1$:

Cor. A $n \times n$, $\|A\| = 1$. Then the following are equiv.:

(a) $\|A^{n-1}\| = 1$ & $w(A) = \cos(\pi/(n+1))$,

(b) $A \cong J_n$,

(c) $\|A^{n-1}\| = 1$ & $A^n = 0$,

(d) $W(A) = \{z \in \mathbb{C} : |z| \leq \cos(\pi/(n+1))\}$ (Wu, 1998).

References:

(I) Anderson:

[1] H.-L. Gau & P.Y. Wu, Companion matrices: reducibility, numerical range and similarity to contractions, Linear Algebra Appl., 383 (2004), 127–142.

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[4] P.Y. Wu, Numerical ranges as circular discs, Applied Math. Lett., 24 (2011), 2115–2117.

(II) Holbrook:

[1] H.-L. Gau, C.-Y. Huang & P.Y. Wu, Numerical radius inequalities for square-zero and idempotent operators, Oper. Matrices, 2 (2008), 137–141.

[2] P.Y. Wu, H.-L. Gau, & M.C. Tsai, Numerical radius inequality for C_0 contractions, Linear Algebra Appl., 430 (2009), 1509–1516.

(III) Williams & Crimmins:

[1] H.-L. Gau & P.Y. Wu, Inner functions of numerical contractions, Linear Algebra Appl., 430 (2009), 2182–2191.

[2] H.-L. Gau, K.-Z. Wang & P.Y. Wu, Numerical radii for tensor products of matrices, 2013, preprint.

國科會補助計畫衍生研發成果推廣資料表

國科會補助計畫	計畫名稱: 張量積的數值域
	計畫主持人: 吳培元
	計畫編號: 101-2115-M-009-004-　　　　學門領域: 泛函分析

<div align="center">

無研發成果推廣資料

</div>

101 年度專題研究計畫研究成果彙整表

計畫主持人：吳培元　　　計畫編號：101-2115-M-009-004-

計畫名稱：張量積的數值域

成果項目			量化			單位	備註（質化說明：如數個計畫共同成果、成果列為該期刊之封面故事...等）
			實際已達成數（被接受或已發表）	預期總達成數(含實際已達成數)	本計畫實際貢獻百分比		
國內	論文著作	期刊論文	0	0	100%	篇	
		研究報告/技術報告	0	0	100%		
		研討會論文	0	0	100%		
		專書	0	0	100%		
	專利	申請中件數	0	0	100%	件	
		已獲得件數	0	0	100%		
	技術移轉	件數	0	0	100%	件	
		權利金	0	0	100%	千元	
	參與計畫人力（本國籍）	碩士生	0	0	100%	人次	
		博士生	1	1	100%		
		博士後研究員	0	0	100%		
		專任助理	0	0	100%		
國外	論文著作	期刊論文	3	3	100%	篇	
		研究報告/技術報告	0	0	100%		
		研討會論文	0	0	100%		
		專書	0	0	100%	章/本	
	專利	申請中件數	0	0	100%	件	
		已獲得件數	0	0	100%		
	技術移轉	件數	0	0	100%	件	
		權利金	0	0	100%	千元	
	參與計畫人力（外國籍）	碩士生	0	0	100%	人次	
		博士生	0	0	100%		
		博士後研究員	0	0	100%		
		專任助理	0	0	100%		

	成果項目	量化	名稱或內容性質簡述
其他成果 (無法以量化表達之成果如辦理學術活動、獲得獎項、重要國際合作、研究成果國際影響力及其他協助產業技術發展之具體效益事項等,請以文字敘述填列。)	無		

	成果項目	量化	名稱或內容性質簡述
科教處計畫加填項目	測驗工具(含質性與量性)	0	
	課程/模組	0	
	電腦及網路系統或工具	0	
	教材	0	
	舉辦之活動/競賽	0	
	研討會/工作坊	0	
	電子報、網站	0	
	計畫成果推廣之參與（閱聽）人數	0	

國科會補助專題研究計畫成果報告自評表

請就研究內容與原計畫相符程度、達成預期目標情況、研究成果之學術或應用價值（簡要敘述成果所代表之意義、價值、影響或進一步發展之可能性）、是否適合在學術期刊發表或申請專利、主要發現或其他有關價值等，作一綜合評估。

1. 請就研究內容與原計畫相符程度、達成預期目標情況作一綜合評估 ■達成目標 □未達成目標（請說明，以 100 字為限） 　　　□實驗失敗 　　　□因故實驗中斷 　　　□其他原因 　說明：
2. 研究成果在學術期刊發表或申請專利等情形： 論文：■已發表 □未發表之文稿 □撰寫中 □無 專利：□已獲得 □申請中 ■無 技轉：□已技轉 □洽談中 ■無 其他：（以 100 字為限）
3. 請依學術成就、技術創新、社會影響等方面，評估研究成果之學術或應用價值（簡要敘述成果所代表之意義、價值、影響或進一步發展之可能性）（以 500 字為限） 無