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Anisotropic perturbation of de Sitter space

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Abstract. A model-independent expression for the Friedmann equation in Bianchi type spaces is derived. In addition, a model-independent stability analysis of the higher curvature de Sitter solution is discussed. Stability conditions of the de Sitter solution are derived explicitly for a cubic model with interesting effects. It is known that quadratic terms do not contribute to this de Sitter background solution. Higher curvature terms are all critical to the stability of the de Sitter space.

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1 Introduction

Our universe is known to be homogeneous and isotropic [1, 2]. Such an universe is described by the well-known Friedmann–Robertson–Walker (FRW) metric [3–6]. There have been, however, some cosmological problems associated with the standard big bang model responsible for the evolution of our present universe. Inflationary models provide resolutions to these problems [7–10]. It is therefore important to find out whether the de Sitter background is a stable final state for any candidate model.

Higher curvature terms should be relevant to the stability of the inflationary physics at the high energy region [11, 12]. Higher curvature terms are effective theories as quantum corrections of matter fields [13–16]. Therefore, the higher curvature effect on the stability of inflation deserves more attention [17–19]. In order to survey possible constraints on the existing models, a model-independent method has been very helpful for the stability analysis of pure gravity theories [14–16, 20, 21].

The stability problem has been discussed for general relativity with a scalar field [22]. Pure gravity models with quadratic curvature terms have also been discussed previously [23–27]. Solutions that do not approach a de Sitter space were found in [23–25]. Instead, we will focus on the stability of de Sitter space against anisotropic perturbations. For simplicity, we will also focus on the stability problem of higher curvature theory with a scalar field. A cubic curvature model will be presented as a simple demonstration. Quadratic terms are known to be irrelevant to the de Sitter solution expansion scale ($H = H_0$) in de Sitter space for pure gravity theories. These quadratic terms are, however, important to the stability of the de Sitter space.

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Note that anisotropic perturbation equations of FRW space are identical to the perturbation equation of anisotropic Bianchi spaces. In addition, relative equations are similar for all Bianchi spaces [14–16, 20, 21]. The latest observation also indicates that the physical universe is a flat space. Therefore, we will focus on the perturbation equation of Bianchi type I (BI) space in this paper.

Field equations will be derived in Sect. 2. The perturbation equation and model-independent stability conditions will be shown in Sect. 3. In Sect. 4, we will focus on the effect of a model with both cubic and quadratic curvature terms. Finally, we will draw some conclusions.

2 Field equations in BI space

The latest observation indicates that our universe is close to the flat FRW space. Therefore, we will focus on the stability analysis of flat FRW space. A canonical derivation of the Einstein equations for a quadratic model is shown in the appendix. In fact, there is an alternative approach to derive the complete set of field equations by treating the system as a constrained system. Indeed, we are interesting in the field equation in the presence of BI space with the following metric:

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a_1^2(t)\,\mathrm{d}x^2 + a_2^2(t)\,\mathrm{d}y^2 + a_3^2(t)\,\mathrm{d}z^2. \qquad (1)$$

The isotropic limit of this space with $a_1 = a_2 = a_3$ is the flat FRW space. Note that this is a constrained system with $g_{ab} = g_{ab}(a_i)$, as shown above. Therefore, the field equations can be derived from the variational δa_i equation as a constrained system via $\delta g_{ab} = (\delta g_{ab}/\delta a_i)\delta a_i$. The only nontrivial thing is that the δg_{00} equation, known as the Friedmann equation, can only be derived from this approach if the lapse function b^2 (in $\mathrm{d}t^2 = b^2(t')\,\mathrm{d}t'^2$) is restored explicitly in the above metric. The lapse function is a cyclic variable with hidden information. In fact, it is known that the Friedmann equation has a smaller differentiation order than the other δa_i equation. This is why the Friedmann equation is known as a constraint and nonredundant equation.

In order to study the anisotropic perturbations, the Friedmann equation in BI space will be adopted. Generalizations to different anisotropic spaces are straightforward. In fact, it turns out that perturbation equations are identical for all Bianchi spaces when we take the de Sitter space as the background space. All nonvanishing components of the curvature tensor can be shown to be [28]

$$R^{ti}_{ti} = \dot{H}_i + H_i^2, \tag{2}$$

$$R^{ij}_{ij} = H_i H_j, \tag{3}$$

for all i, j in cyclic order and its proper permutations. Here $H_i \equiv \dot{a}_i / a_i$.

The Friedmann equation and the δa_i equations of the pure gravity model L can be shown to be [28]

$$DL \equiv \mathcal{L} + H_i \left(\frac{\mathrm{d}}{\mathrm{d}t} + 3H \right) L^i - H_i L_i - \dot{H}_i L^i = 0\,, \tag{4}$$

$$D_i L \equiv \mathcal{L} + \left(\frac{\mathrm{d}}{\mathrm{d}t} + 3H \right)^2 L^i - \left(\frac{\mathrm{d}}{\mathrm{d}t} + 3H \right) L_i = 0\,. \tag{5}$$

Here we have defined the reduced Lagrangian $L = \sqrt{g}\mathcal{L} = L(a_i(t))$ of a pure gravity model in BI spaces by

$$L = V\mathcal{L}\left(R^{ti}_{tj}, R^{ij}_{kl} \right) = V\mathcal{L}\left(H_i, \dot{H}_i \right), \tag{6}$$

where $V \equiv a_1 a_2 a_3$ is the volume measure of the BI space. In addition, $L_i \equiv \delta\mathcal{L}/\delta H_i$, $L^i \equiv \delta\mathcal{L}/\delta\dot{H}_i$, and $3H \equiv \sum_i H_i$. The Bianchi identity shows that the perturbation equation associated with the a_i equation becomes redundant in the de Sitter background. Also, for convenience, $\mathcal{L}$ will be written as L from now on.

The Friedmann equation shown above is in fact a universal formula, which holds for all Bianchi type spaces. Indeed, the lapse function b^2 in the metric $\mathrm{d}s^2 = -b^2(t)\,\mathrm{d}t^2 + g_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j$ can be chosen as $b = 1$ by a redefinition of t for convenience. b is known, however, to be a cyclic variable that hides the nonredundant Friedmann equation $G^{tt} = T^{tt}$ as a nontrivial constraint of the system. The Friedmann equation is known to be nonredundant following the Bianchi identity. Therefore, a compact formula for the Friedmann equation may serve as a better tool for a model-independent analysis of any gravitational system.

Fortunately, we can always derive the hidden Friedmann equation by the variational principle with respect to δb, or equivalently δB, once the cyclic variable b^2 is restored in the effective Lagrangian L. In order to derive a model-independent formula for the Friedmann equation in terms of the variables H_i and $\dot{H}_i$, we need to replace the effect of $\delta L/\delta B$ and $\delta L/\delta\dot{B}$ as an equivalent formula depending only on $\delta L/\delta H_i$ and $\delta L/\delta\dot{H}_i$. The proof follows from the observation that $\dot{B}$ always shows up as a combination of $\dot{B}H_i + 2B(\dot{H}_i + H_i^2)$ or BH_iH_j in the Lagrangian of all Bianchi type spaces when the lapse function $b^2(t) \equiv 1/B^2(t)$ is restored. Explicitly, $\delta L/\delta\dot{B} = H_i \delta L/[2\delta\dot{H}_i]$. Here we have set $B = 1$ whenever it will not affect the final result. Moreover, the summation over repeated indices is not written explicitly. In addition $\delta L/\delta B = H_i \delta L/[2\delta H_i] + \dot{H}_i \delta L/\delta\dot{H}_i$ if $L = L(B(a^i\dot{H}_i + a^{ij}H_iH_j))$ for arbitrary "constant" coefficients a^i and a^{ij}. In fact, $B\dot{H}_i$ will always show up together with $B_1 H_i H_j$, as can be seen from a dimensional analysis. Therefore the Friedmann equation derived above is a universal formula for all Bianchi spaces.

3 Higher derivative gravity model with a scalar field

With a scalar field (with Lagrangian L_ϕ) coupled to the pure gravity Lagrangian L_g, we have

$$L = L_g + L_\phi \equiv L_g\left(H_i, \dot{H}_i \right) - \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi)\,, \tag{7}$$

with $V(\phi)$ the scalar potential of the scalar field. The Friedmann equation can be written as

$$DL_g = \frac{1}{2}\dot{\phi}^2 + V(\phi). \tag{8}$$

In addition, the scalar field equation can be shown to be

$$\ddot{\phi} + 3H\dot{\phi} + V' = 0. \tag{9}$$

We will focus on the stability of an inflationary de Sitter background solution characterized by a constant Hubble parameter $H_i = H_0$ in addition to a slow roll-over scalar field ϕ. Equivalently, we will write $H_i = H_0 + \delta H_i$ and $\phi = \phi_0 + \delta\phi$ as the anisotropic perturbation against the de Sitter background space. Consequently, we have a set of zeroth order equations:

$$DL_g(H_i = H_0) = V(\phi_0)\,, \tag{10}$$

$$V'(\phi_0) = 0. \tag{11}$$

The constraint $V'(\phi_0) = 0$ can be realized at two different stages: (i) in the inflationary phase where $\phi = 0$ as a local maximum of some SSB potential V, (ii) in the final state where $\phi = \phi_m$ approaches the local minimum of V. A model with a specific V will be shown shortly. This final state is expected to be a stable vacuum. As a result, the following stability equation can be derived from perturbing DL_g defined in (4):

$$\begin{aligned} \delta(DL_g) = &\left\langle H_i L^{ij} \delta\ddot{H}_j \right\rangle + 3H \left\langle H_i L^{ij} \delta\dot{H}_j \right\rangle \\ &+ 3H \left\langle \left(H_i L^i_j + L^j \right) \delta H_j \right\rangle \\ &+ \left\langle H_i L^i \right\rangle \delta(3H) - \left\langle H_i L_{ij} \delta H_j \right\rangle. \end{aligned} \tag{12}$$

Here $H \equiv \sum_i H_i/3 = \dot{V}/(3V)$. In addition, the notation $A_iB_i \equiv \langle A_iB_i \rangle \equiv \sum_{i=1}^{3} A_iB_i$ is for the summation over $i = 1$ to 3 for repeated dummy indices. In addition, $L^i_j \equiv \delta^2 L_g/\delta\dot{H}_i\delta H_j$ and similarly for L_{ij} and L^{ij} with upper index ^{i} and lower index $_j$ denoting variation with respect to $\dot{H}_i$ and H_j, respectively.

Defining

$$\mathcal{D}_g \delta H \equiv H_0 \Big[L_{02} \delta \ddot{H} + 3H_0 L_{02} \delta \dot{H} + (6L_{01} + 3H_0 L_{11} - L_{20}) \delta H \Big] \,, \quad (13)$$

the stability equation of (8), with $H_i = H_0 + \delta H_i$ and $\phi = \phi_0 + \delta\phi$, can be shown to be

$$\mathcal{D}_g \delta H = V'(\phi_0) \delta\phi = 0 \,. \quad (14)$$

Here

$$L_{ab} \equiv \delta^{a+b} L / \delta H_{i_i} \delta H_{i_2} \cdots H_{i_a} \delta \dot{H}_{j_1} \delta \dot{H}_{j_2} \cdots \delta \dot{H}_{j_b} |_{H_i \to H_0} .$$

Note that, as claimed earlier, this equation is exactly the same as the isotropic perturbation equation of the flat FRW solution [14–16, 20, 21]. Therefore, (14) becomes

$$\delta\ddot{H} + 3H_0 \delta\dot{H} + K H_0^2 \delta H = 0 \,, \quad (15)$$

with

$$K \equiv \frac{6L_{01} + 3H_0 L_{11} - L_{20}}{L_{02} H_0^2} \,,$$

in BI space. An explicit expression for K can be derived for any given model. As a result, the values of K are critical to the stability of the corresponding de Sitter universe. General selection rules can hence be obtained in a straightforward way. We will focus on the cubic models in the following section for a simple demonstration. We will discuss the model-independent stability conditions for the de Sitter background in this section.

Similarly, the perturbation of the scalar field equation (9) can be shown to be

$$\delta\ddot{\phi} + 3H_0 \delta\dot{\phi} + V_0'' \delta\phi = 0 \,. \quad (16)$$

In fact, the scalar field equation can be solved in the de Sitter background $H_i = H_0$ and $V_0' \equiv V'(\phi_0) = 0$. Indeed, the solution to the equation $\ddot{\phi} + 3H_0\dot{\phi} \sim 0$ is

$$\phi \sim \phi_0 + \frac{\dot{\phi}_0}{3H_0} [1 - \exp(-3H_0 t)] \,. \quad (17)$$

This result indicates that the scalar field does change very slowly. Such a behavior is exactly identical to the behavior of a slow roll-over scalar field.

Assuming that $\delta H = \exp[hH_0 t] \delta H_0$ and $\delta\phi = \exp[pH_0 t] \delta\phi_0$ for some constants h and p, one can write the above equations as

$$(h^2 + 3h + K) \delta H = 0, \quad (18)$$

$$\left(p^2 + 3p + \frac{V_0''}{H_0^2} \right) \delta\phi = 0. \quad (19)$$

Note that the δH equation is the same as the pure gravity model, independent of the scalar field. The effect of the scalar field is minor both in the inflationary phase and the final stage. Therefore, the de Sitter space can hopefully be a stable background in both stages with $V'(\phi) \sim 0$. This is a positive sign: a stable de Sitter space as a final state is what we need. The difference between these two stages is that ϕ cannot stay constant forever when the initial ϕ is close to the local maximum of V. ϕ will slide off the local maximum according to the slow roll-over equation (17). Therefore, the inflationary phase is not a stable state for ϕ. On the other hand, ϕ will oscillate with a damping term around the local minimum of V and eventually settle down to the local minimum. Therefore, the final state is a stable final state for ϕ.

As a result, we can have a stable mode for δH in both states of ϕ (at $V' = 0$) with a similar structure given by the stability condition (18). The only difference is that $H = H_0$ in the inflationary phase and $H = H_m$ in the final state. Here H_0 and H_m are both constants characterized the Hubble expansion scale of these different states. Hence inflation will be ended once the scalar field rolls off the initial phase $V_0' = 0$. When it rolls down to the local minimum, $V'(\phi_m) = 0$, the evolution of the de Sitter solution will be similar to the inflationary phase solution discussed here.

Indeed, (18) and (19) indicate that there are two decaying modes for δH and $\delta\phi$ with

$$2h = -3 \pm \sqrt{9 - 4K}, \quad (20)$$

$$2p = -3 \pm \sqrt{9 - 4V''(0)/H_0^2} \,. \quad (21)$$

We need at least a stable δH solution with negative h, so that inflation is possible along this stable direction. It will be even better if both h solutions are negative. In addition, we need at least one unstable solution requiring either p or h to be positive. This unstable mode will end the inflationary phase automatically.

Explicitly, $K > 0$ will make both h solutions negative. $K > 9/4$ will make $\sqrt{9 - 4K}$ imaginary and hence turns $\exp[hH_0 t] = \exp[-3H_0 t] \cos\left[\sqrt{4K - 9} H_0 t + \theta_1\right]$ into an oscillatory solution with a constant phase θ_1. Note that this will also make δH a stable mode. On the other hand, we will have a stable mode and an unstable mode if $K < 0$. The case $K = 0$ gives us one negative and one zero h solution. Both ($K = 0$) modes are stable again. In summary, the condition $K \geq 0$ implies two stable modes of δH. One stable mode of δH is enough for inducing inflation. But the two stable modes of δH will further ensure that the anisotropy will not grow out of control in this model.

An appropriate effective spontaneously symmetry breaking potential V of the following form:

$$V(\phi) = \frac{\lambda}{4} (\phi^2 - v^2)^2 + V_m \,, \quad (22)$$

with arbitrary coupling constant λ, can be shown to be a good candidate of such models. Here V_m is a small cosmological constant dressing the SSB potential. When the scalar field eventually rolls down to the minimum of V at $\phi = v$, the system will oscillate around this local minimum with a friction term related to the effective Hubble constant H_m at this stage. A reheating process is expected to take away the kinetic energy of the scalar field. The scalar field will eventually become a constant background field and lose all its kinetic energy.

H_0 can be chosen to induce enough inflation for a brief moment as long as the slow roll-over scalar field remains close to the initial state $\phi = \phi_0$. This de Sitter phase will hence remain stable and drive the inflationary process for a brief moment. Explicit models with a cubic coupling term will be discussed as an example in the next section.

Explicitly, the solutions for p are $p = p_\pm = -3/2 \pm \sqrt{9 - 4V_0''/H_0^2}/2$. Hence $p_\pm = -3/2 \pm \sqrt{9 + 4\lambda v^2/H_0^2}/2$ for the SSB ϕ^4 potential model. Therefore, it is easy to find that the solutions $p_+ > 0$ (unstable mode) and $p_- < 0$ (stable mode) exist for this model. In addition, with properly chosen parameters, the unstable mode $p_+ = \sqrt{9 + 4\lambda v^2/H_0^2}/2 - 3/2$ can be made small enough to induce 60 e-folds of inflation during the inflationary phase. When $\phi \to v$, the scalar field will remain a stable mode perturbatively. The final de Sitter space will remain stable against anisotropic perturbations as long as $K(H_m) \geq 0$ accommodates two stable modes.

4 Cubic model

In this section, we will study the higher derivative gravity model with a coupled scalar field:

$$\begin{aligned}\mathcal{L} &= -R - \alpha R^2 - \beta R^\mu_\nu R^\nu_\mu + \gamma R^{\mu\nu}_{\beta\gamma} R^{\beta\gamma}_{\sigma\rho} R^{\sigma\rho}_{\mu\nu} \\ &\quad - \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi) \\ &\equiv L_g + L_\phi \,. \end{aligned} \tag{23}$$

Here $L_g = -R - \alpha R^2 - \beta R^\mu_\nu R^\nu_\mu + \gamma R^{\mu\nu}_{\beta\gamma} R^{\beta\gamma}_{\sigma\rho} R^{\sigma\rho}_{\mu\nu}$ and $L_\phi = -\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi)$ denote the pure gravity Lagrangian and scalar field Lagrangian, respectively. We will also write $L_1 = -R$, $L_2 = -\alpha R^2 - \beta R^\mu_\nu R^\nu_\mu$ and $L_3 = \gamma R^{\mu\nu}_{\beta\gamma} R^{\beta\gamma}_{\sigma\rho} R^{\sigma\rho}_{\mu\nu}$ for convenience. The cubic term is shown to be the two-loop effect of super gravity [14–16]. Note also that this is the most general covariant quadratic gravity model. The quadratic term $R^{ab}_{cd} R^{cd}_{ab}$ is related to the α and β terms by the Euler invariant.

In a moment, we will show that quadratic terms (1) do not contribute to the expanding parameter H_0, and (2) will affect the stability of the de Sitter phase [28]. We can write the Lagrangian (23) explicitly as

$$\begin{aligned} L &= 6\left(\dot{H} + 2H^2\right) - 36\alpha\left[\dot{H} + 2H^2\right]^2 \\ &\quad - 12\beta\left[\dot{H}^2 + 3\dot{H}H^2 + 3H^4\right] \\ &\quad + 24\gamma\left[(\dot{H} + H^2)^3 + H^6\right] , \end{aligned} \tag{24}$$

when we set $H_i \to H$. The leading order equation of Friedmann equation reads

$$DL_g(H_i = H_0) = V(\phi_0) \equiv V_0 \,, \tag{25}$$

in the de Sitter background with $H_i = H_0$ and $\phi = \phi_0$. Explicitly, we have

$$V_0 = 6\left[1 - 4\gamma H_0^4\right] H_0^2 \,. \tag{26}$$

Note that quadratic terms do not contribute to H_0 as promised earlier. This result is a general property associated with the conformal structure of de Sitter space. In fact, this result also follows from the fact that $a^3(\dot{H} + H^2)H^2 = \mathrm{d}[a^3 H^3]/[3\,\mathrm{d}t]$ is a total derivative. Therefore, $(\dot{H} + H^2)H^2$ will not affect the field equation. Hence the only effects of the quadratic terms come from the remaining Lagrangian $L_2' = -12(3\alpha + \beta)\dot{H}^2$. This term will contribute to (25) in the de Sitter background. As a result, quadratic terms will not affect the expansion scale H_0. Quadratic terms will, however, affect the linear order perturbation equation and consequently the stability of de Sitter solution. Note that when $\gamma = 0$, (26) implies that $V_0 = 6H_0^2$. None of the quadratic terms is affecting the expansion rate H_0. This indicates that the γ term does affect the expansion rate in a very complicated way. Fortunately, (26) can be solved and served as an useful tool in the forthcoming analysis.

In addition, the coefficient K for the δH perturbation equation can be shown to be

$$K_0 = \frac{1 - 12\gamma H_0^4}{2H_0^2\left[6\gamma H_0^2 - 3\alpha - \beta\right]} \tag{27}$$

for this model, which has two different decaying modes $h = [-3 \pm \sqrt{9 - 4K}]/2$.

Writing $x = H_0^2$, the polynomial equation (26) can be solved to give $x = x_1$ and $x = x_\pm$ with

$$x_1 = -\left(\frac{1}{3\gamma}\right)^{1/2} \cos\frac{\theta_0}{3}, \tag{28}$$

$$x_\pm = \left(\frac{1}{3\gamma}\right)^{1/2} \cos\frac{\theta_0 \mp \pi}{3}. \tag{29}$$

Here $\cos\theta_0 \equiv \sqrt{3\gamma}V_0/2 \leq 1$. The notation $x_\pm$ is defined such that $x_- \leq x_+$. In addition, these two solutions become degenerate when $3\gamma V_0^2 = 4$.

Similarly, when the system settles close to the final de Sitter phase at $\phi \to v$, similar solutions hold for this state. Explicitly, we have

$$K_m = \frac{1 - 12\gamma H_m^4}{2H_m^2\left[6\gamma H_m^2 - 3\alpha - \beta\right]} \,, \tag{30}$$

for this model, which has two different decaying modes $h_m = \left[-3 \pm \sqrt{9 - 4K_m}\right]/2$.

Writing $y = H_m^2$, the polynomial equation (26) can be solved to give $y = y_1$ and $y = y_\pm$ with

$$y_1 = -\left(\frac{1}{3\gamma}\right)^{1/2} \cos\frac{\theta_m}{3} \,, \tag{31}$$

$$y_\pm = \left(\frac{1}{3\gamma}\right)^{1/2} \cos\frac{\theta_m \mp \pi}{3} \,. \tag{32}$$

Here $\cos\theta_m \equiv \sqrt{3\gamma}V_m/2 \le 1$. The notation $y_\pm$ is defined such that $y_- \le y_+$. In addition, these two solutions become degenerate when $3\gamma V_m^2 = 4$.

Different regions of physical solutions are shown in Fig. 1. Indeed, a physical solution exists for $-\pi/2 \le \theta_0 \le \pi/2$. As a result, we have $\pi/6 \le (\theta_0+\pi)/3 \le \pi/2$ when $x = x_+$. There is another set of solutions for $x = x_-$ with $-\pi/2 \le (\theta_0-\pi)/3 \le -\pi/6$. There are also two similar sets of solutions for y. Note that $V_m < V_0$ implies that $|\theta_m| > |\theta_0|$. In addition, $H_0 \gg H_m$ is a physical assumption of the inflationary model. Therefore, the only way to get a physical solution for $H_0^2 \gg H_m^2$ is to take $x = x_+$ and $y = y_- \to 0$. This can be shown by a simple observation; see (29) and (32). The only way to make H_m small as compared to H_0 and to make θ_m obey $|\theta_m| > |\theta_0|$ is to choose $(\theta_m - \pi)/3 \to 0$. This can be done by writing $(\theta_m - \pi)/3 = \epsilon_m - \pi/2$ for some small constant ϵ_m. Consequently, we have $\theta_m = -\pi/2 + 3\epsilon_m$ and

$$H_m^2 = \left(\frac{1}{3\gamma}\right)^{1/2} \cos\frac{\theta_m - \pi}{3} = \left(\frac{1}{3\gamma}\right)^{1/2} \cos\left(-\frac{\pi}{2} + \epsilon_m\right) \sim \left(\frac{1}{3\gamma}\right)^{1/2} \epsilon_m \,. \tag{33}$$

This result shows that $12\gamma H_m^4 \sim \epsilon_m^2$. In addition, we can either write $(\theta_0+\pi)/3 = \pi/6 + \epsilon_0$ or $(\theta_0 - \pi)/3 = -\pi/6 - \epsilon_0$ for some small constant ϵ_0. This is equivalent to chosing $\theta_0 = -\pi/2 + 3\epsilon_0$ or $\theta_0 = \pi/2 - 3\epsilon_0$. This will make $|\theta_m| > |\theta_0|$ and will make H_0 as big as possible. As a result, we have

$$H_0^2 = \left(\frac{1}{3\gamma}\right)^{1/2} \cos\frac{\theta_0 \pm \pi}{3} = \left(\frac{1}{3\gamma}\right)^{1/2} \cos\left(-\frac{\pi}{6} \pm \epsilon_0\right) \sim \left(\frac{1}{3\gamma}\right)^{1/2} \left(\frac{1}{2} \mp \frac{\sqrt{3}}{2}\epsilon_0\right) . \tag{34}$$

The result shows that $12\gamma H_0^4 \sim 1 \mp 2\sqrt{3}\epsilon_0$. In summary, we have

$$12\gamma H_0^4 \sim 1 \mp 2\sqrt{3}\epsilon_0, \tag{35}$$

$$12\gamma H_m^4 \sim \epsilon_m^2 \,, \tag{36}$$

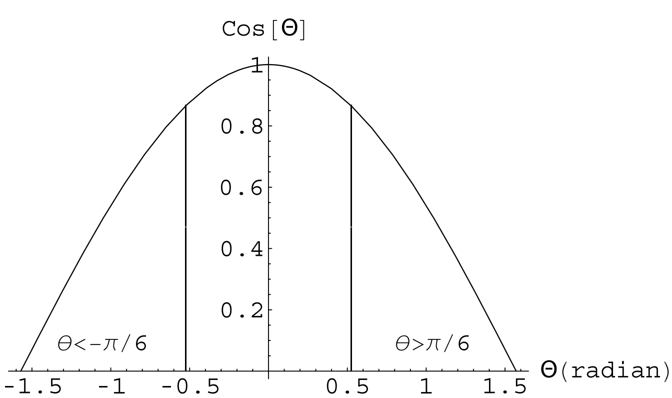

Fig. 1. $\cos\theta$ is plotted with two ranges $\theta > \pi/6$ and $\theta < -\pi/6$ specified. $(\theta_m - \pi)/3$ is expected to be close to $-\pi/2$

when the coupling constants are properly chosen. Here we have assumed that $\epsilon_0 \ll 1$ and $\epsilon_m \ll 1$ in order to observe the physical pattern of these solutions.

As shown earlier, writing $\delta H = \exp[hH_0t]\delta H_0$ and $\delta\phi = \exp[pH_0t]\delta\phi_0$ for some constants h and p, the perturbation equations (18)–(19) become

$$h^2 + 3h + \frac{1 - 12\gamma H_0^4}{2H_0^2\left[6\gamma H_0^2 - 3\alpha - \beta\right]} = 0 \,, \tag{37}$$

$$\left(p^2 + 3p + \frac{V_0''}{H_0^2}\right) = 0 \,. \tag{38}$$

As a result, the solution to the equation for h (37) is $h = h_\pm = -3(1 \pm \delta_2)/2$ with

$$\delta_2^2 = 1 + 2\frac{\left(12\gamma H_0^4 - 1\right)}{\left\{9H_0^2\left[6\gamma H_0^2 - 3\alpha - \beta\right]\right\}} \,.$$

In addition, the solution to the equation for p (38) is $p = p_\pm = -3/2 \pm \sqrt{9 - 4V_0''/H_0^2}/2 = -3/2 \pm \sqrt{9 + 4\lambda v^2/H_0^2}/2$ for the SSB ϕ^4 potential model. Here $p_+ > 0$ and $p_- < 0$ indicate an unstable mode and a stable mode for this model. Properly chosen coupling constants α, β and γ allow the unstable p-mode to have a long enough Δt in the inflationary phase. As a result, inflation of 60 e-folds can be induced. Indeed, this is the amount to require that $p_+ \le 1/60$, or equivalently, $\lambda v^2 \sim 0.052\bar{7}H_0^2$.

We can also write the perturbative solution for H_i and ϕ as

$$H_i = H_0 + A_{i+}\exp[h_+H_0t] + A_{i-}\exp[h_-H_0t] \,, \tag{39}$$

$$\phi = \phi_0 + B_{i+}\exp[p_+H_0t] + B_{i-}\exp[p_-H_0t] \,, \tag{40}$$

with $A_{i\pm}$ and $B_{i\pm}$ some constant coefficients determined by the initial perturbations. These linear solutions become oscillatory solutions if the discriminant δ_2 becomes pure imaginary. In such case, these equations take the following form:

$$H_i = H_0 + A_i\exp[-3H_0t/2]\cos[3|\delta_2|H_0t/2 + \theta_{A_i}] \,, \tag{41}$$

with A_i and θ_{A_i} some constant coefficients.

We have shown that the h perturbation have two stable modes only when $K \ge 0$. Explicitly, these inequalities will hold when either

$$(1) \quad 2(3\alpha+\beta)H_0^2 < 12\gamma H_0^4 < 1 \,, \tag{42}$$

$$(2) \quad 1 < 12\gamma H_0^4 < 2(3\alpha+\beta)H_0^2 \,, \tag{43}$$

holds. Consequently, the de Sitter background can remain stable with properly chosen coupling constants α, β and γ. Note that the above inequalities imply that the L_1, L_2 and L_3 Lagrangians are competing for physical solutions. For example, condition (1) in the above equation states that "$L_1 > L_3 > L_2$". Therefore, coupling constants have to be chosen carefully to accommodate a physical solution.

Note that similar solutions for p and h also exist when the scalar field rolls down the local minimum of the SSB potential:

$$(1) \quad 2(3\alpha+\beta)H_m^2 < 12\gamma H_m^4 < 1 \,, \tag{44}$$

$$(2) \quad 1 < 12\gamma H_m^4 < 2(3\alpha+\beta)H_m^2 \,. \tag{45}$$

The second set of solutions of H_m is clearly inconsistent with (36). Therefore the only consistent H_m solution is (44). In summary, a consistent solution exists only when

$$2(3\alpha+\beta)H_0^2 < 12\gamma H_0^4 < 1\,, \tag{46}$$
$$2(3\alpha+\beta)H_m^2 < 12\gamma H_m^4 < 1\,, \tag{47}$$

or

$$2(3\alpha+\beta)H_m^2 < 12\gamma H_m^4 < 1 < 12\gamma H_0^4 < 2(3\alpha+\beta)H_0^2\,, \tag{48}$$

hold separately. Equation (26) implies that $1 > 4\gamma H_0^4 \gg 4\gamma H_m^4$. Therefore the set of solutions (46)–(47) imply that

$$2(3\alpha+\beta)H_m^2 < \left(12\gamma H_m^4\right) < 2(3\alpha+\beta)H_0^2 < \left(12\gamma H_m^4\right) < 12\gamma H_0^4 < 1. \tag{49}$$

Here the term $\left(12\gamma H_m^4\right)$ can be in either position for consistency. Consequently, physical solutions that approach the de Sitter space as a final stable state can be found in this model.

Consider the special case where $\gamma = 0$; then the condition for a stable de Sitter final state is

$$2(3\alpha+\beta)H_m^2 < 2(3\alpha+\beta)H_0^2 < 1, \quad \text{or} \tag{50}$$
$$2(3\alpha+\beta)H_m^2 < 1 < 2(3\alpha+\beta)H_0^2\,. \tag{51}$$

Similarly, for the case $\alpha = \beta = 0$, the condition for a stable de Sitter final state is

$$12\gamma H_m^4 < 12\gamma H_0^4 < 1, \tag{52}$$
$$\text{or} \quad 12\gamma H_m^4 < 1 < 12\gamma H_0^4\,. \tag{53}$$

In addition, there is the special case when $L_{02} = 2H_0^2(6\gamma H_0^2 - 3\alpha - \beta)$ or $L_{02} = 2H_m^2(6\gamma H_m^2 - 3\alpha - \beta)$. In such cases, the perturbative equation for h becomes $\delta H = 0$. This means that the corresponding de Sitter solution is absolutely stable against any anisotropic perturbation.

In summary, the presence of a scalar field makes the system far more complicated than the system without scalar field, especially in higher curvature gravity models. A scalar field can take care of the ending of the inflationary phase in a natural way. It also introduces a strong constraint on the system. Fortunately, a consistent and physical solution can always be found to support the de Sitter space as a stable final state.

5 Conclusion

The existence of a stable de Sitter background is closely related to the choices of the coupling constants. We have shown that, for gravity models with an additional scalar field, the flat FRW de Sitter background space can be a background if the coupling constants are chosen properly. The ending of the inflationary process is due to the unstable mode of a slow roll-over scalar field with a SSB potential.

An explicit model with a spontaneously symmetry breaking ϕ^4 potential is presented as a simple demonstration. It is also shown explicitly that quadratic terms will not affect the de Sitter solution characterized by the Hubble parameters H_0 and H_m. In particular, the simple observation that the effective quadratic Lagrangian $L_2' = -12(3\alpha+\beta)\dot{H}^2$ has been shown explicitly. Quadratic terms play, however, a critical role in the stability of the de Sitter background. Indeed, with properly chosen coupling constants, the anisotropy can only grow mildly. Implications of these stability conditions deserve more attention in the search for physical models.

Appendix: Field equations

The field equation of the Lagrangian $L = -R - \alpha R^2 - \beta\left(R^a_b\right)^2 - \partial_a\phi\partial^a\phi/2 - V$ can be derived by the variation of g_{ab}. The result is

$$\frac{1}{2}Rg_{ab} - R_{ab} + \frac{1}{2}g_{ab}\left[\alpha R^2 + \beta(R^c_d)^2 - L_\phi\right] = 2\left(\alpha R\, R_{ab} + \beta R_{ac}R^c_b\right) - 2\alpha\left(g_{ab}D^2 - D_aD_b\right)R - \frac{\beta}{2}g_{ab}D^2R - \beta D^2R_{ab} + 2\beta D_aD_cR^c_b + \frac{1}{2}\partial_a\phi\partial_b\phi\,. \tag{A.1}$$

In addition, the scalar equation can be shown to be

$$D^2\phi = V'\,. \tag{A.2}$$

In order to derive the field equation in a covariant way, we may write the variation of the Riemann curvature tensor as $\delta R^d_{cba} = -D_a\delta\Gamma^d_{bc} + D_b\delta\Gamma^d_{ac}$ as if $\delta\Gamma^a_{bc}$ is a type $T(1,2)$ tensor. The derivation has nothing to do with whether $\delta\Gamma^a_{bc}$ is a tensor or not. Rather, by imagining $\delta\Gamma^a_{bc}$ is a tensor and using all related properties of a tensor, it helps in reducing the effort in deriving these equations, especially when integration-by-parts is required. In addition, we have also used the Bianchi identity $D_cD_DR^{acdb} = D^2R^{ab} - D_cD^aR^{bc}$ in converting the differentiation of the Riemann tensor into a differentiation of the Ricci tensor. The field equation of the cubic term can be derived similarly.

In summary, the reduced formulae shown in this paper can be helpful in extracting some useful information without going into the details of the field equations. For example, the existence of the inflationary solution $H = H_0$ has to do with the leading order equations. It can be done by ignoring any term like $f(H)\dot{H}$, with $f(H)$ an arbitrary function of H. On the other hand, the stability of the inflationary solution has to do with those leading order terms linear in the time differentiation of δH. We can freely ignore terms like $\dot{H}^2$. In particular, $(\mathrm{d}/\mathrm{d}t)(f(H)\delta H) = f(H)\delta\dot{H}$ can be used to skip unrelated terms, with $f(H)$ an arbitrary function of H, with the closed formula shown in this paper.

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