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中 文 摘 要 ： 複合選擇權是一種賦予持有方買或賣該標的選擇權權利的商品。在本文裡，我們要探討永續美式複合權在跳躍擴散模型的假設下的評價問題。從 Gapeev 和 Rodosthesnous 的研究可得知，在跳躍擴散模型的假設下，這種起初是雙重最佳停止時間的問題能被分解成一連串的單一最佳停止問題。利用處理單一最佳停止問題常用的平均方法，我們推導出永續美式複合選擇權在雙重指數型跳躍擴散模型下的顯解。

中文關鍵詞： 永續美式複合選擇權、最佳停止問題、雙重指數型跳躍擴散模型

英 文 摘 要 ：

英文關鍵詞：

Pricing perpetual American compound options under a jump-diffusion model

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Abstract

A compound option gives the holder the right to buy or sell the underlying option. In this paper, we consider the pricing problem of perpetual American compound options when the underlying dynamics follow a jump-diffusion process. Following Gapeev and Rodosthenous, the initial two-step optimal stopping problems are decomposed into sequences of one-step problems for the underlying jump-diffusion process. Using the averaging approach for the usual one-step optimal stopping problems, we give explicit solutions to the perpetual American option pricing problems in the double-exponential jump-diffusion model.

1 Introduction

The one-step optimal stopping problems we consider in this paper will be of the form

$$V(x) = \sup_{\tau \in \mathcal{T}} \mathbb{E}_x(e^{-r\tau} g(X_\tau)) \tag{1.1}$$

where $X = \{X_t : t \geq 0\}$ under $\mathbb{P}_x$ is a Lévy process started from $X_0 = x$. Further, g is a measurable function, $r \geq 0$ and $\mathcal{T}$ is a family of stopping times with respect to the natural filtration generated by X, $\mathbb{F} = \{\mathcal{F}_t : t \geq 0\}$. The optimal stopping problem consists of finding the optimal stopping time τ^* such that $V(x) = \sup_{\tau\in\mathcal{T}} \mathbb{E}_x(e^{-r\tau} g(X_\tau)) = \mathbb{E}_x(e^{-r\tau^*} g(X_{\tau^*}))$. Also we need to find the corresponding optimal reward (the value function): $V(x) = \mathbb{E}_x(e^{-r\tau^*} g(X_{\tau^*}))$.

In the literature, there are many different approaches for solving the problem (1.1). In particular, Surya [16] proposed an averaging approach for solving the optimal stopping problem (1.1) in a general setting. His approach does not appeal to a free boundary problem associated to the optimal stopping problem. Instead he first introduced an averaging problem for a given reward function g. Then he showed that an explicit optimal solution can be founded if there is a solution to the averaging problem that has certain monotonicity properties. Recently, when the process X is a jump-diffusion process of the form in (2.8), Sheu and Tsai [15] presented explicit solutions to the averaging problems for a class of the American call type reward functions and hence solved the corresponding optimal stopping problem (1.1).

A compound option is a standard option with another standard option being the underlying asset. There are four basic types of compound options: a call on a call, a put on a call, a call on a put, and a put on a put. Consider, for example, a call on a put of European type. On the first exercise date T_1, the holder of the compound option is entitled to pay the first strike price, K_1, and receive a put option. The put option gives the holder the right to sell the underlying asset for the second strike price, L_2, on the second exercise date, T_2. The compound option will be exercised on the first exercise date only the value of the option on that date is greater than the first strike price. In the Black-Scholes framework, European-style compound options can be valued analytically in terms of integrals of the bivariate normal distribution(see, for example, Geske [7]). For more general underlying dynamics, either explicit solutions do not exist or the integrals become difficult to evaluate. On the other hand, in the literature, many researchers considered the compound options of American type. In Chiarella and Kang [5], the authors formulated the pricing problem for American compound options as the solution to a two-step free boundary problem which is solved numerically via a sparse grid approach. In [6], Gapeev and Rodosthenous considered the pricing problem of perpetual American compound options when the underlying dynamics follow the geometric Brownian motion. By solving the associated sequence of one-sided free boundary problems and the martingale verification, they obtained explicit pricing formulas for all four types of perpetual American compound options. In this paper, we consider the pricing problem of perpetual American compound options when the underlying dynamics follow a jump-diffusion process. Following Gapeev and Rodosthenous [6], the initial two-step optimal stopping problems are decomposed into sequences of one-step problems for the underlying jump-diffusion process. In the double-exponential jump-diffusion model, using the results obtained in Surya [16] and Sheu and Tsai [15], we give explicit solutions for the associated optimal stopping problems and, hence obtain the explicit pricing formula for the perpetual American option pricing problems. By our approach, we also recover results obtained in Gapeev and Rodosthenous [6].

2 Preliminaries

Let $X = \{X_t : t \geq 0\}$ be a real-valued Lévy process defined on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$ such that $X_0 = 0$ a.s. A Lévy process starts from $X_0 = x$ is simply defined as $x + X_t$ for $t \geq 0$ and we denote its law by $\mathbb{P}_x$. Denote by $\mathbb{E}_x$ the expectation with respect to the probability measure $\mathbb{P}_x$. For convenience we shall write $\mathbb{P}$ for $\mathbb{P}_0$ and $\mathbb{E}$ for $\mathbb{E}_0$. The Levy-Khinchine formula states that $\mathbb{E}(e^{\mathrm{i}uX_t}) = e^{t\psi(u)}$, where ψ is called the characteristic exponent of X and is given by the formula

$$\psi(u) = \mathrm{i}au - \frac{1}{2}b^2u^2 + \int_{\mathbb{R}} (e^{\mathrm{i}ux} - 1 - \mathrm{i}ux1_{\{|x|<1\}})\Pi(dx). \tag{2.1}$$

Here $a \in \mathbb{R}$, $b \geq 0$ and Π is a measure on $\mathbb{R}\backslash\{0\}$ such that $\int_{\mathbb{R}} 1 \wedge x^2\Pi(dx) < \infty$.

Throughout this paper, we denote by e_r an exponential random variable with parameter $r > 0$, independent of the process X. In addition, we denote by

$$M_r = \sup_{0\leq s\leq \mathrm{e}_r} X_s \quad \text{and} \quad I_r = \inf_{0\leq s\leq \mathrm{e}_r} X_s,$$

the supremum and the infimum of the Lévy process X killed at the independent exponential random time e_r. Recall the following well-known Wiener-Hopf factorization formula,

$$\mathbb{E}e^{\mathrm{i}uX_{\mathrm{e}_r}} = \frac{r}{r - \psi(u)} = \psi_r^+(u)\psi_r^-(u) \tag{2.2}$$

where $\psi_r^+(u) = \mathbb{E}(e^{\mathrm{i}uM_r})$ and $\psi_r^-(u) = \mathbb{E}(e^{\mathrm{i}uI_r})$.

Theorem 2.1 *Given a reward function g with $H \equiv \{g > 0\} = (\widehat{a}, \infty)$ for some $\widehat{a} < \infty$. Suppose that $\widetilde{Q}_g$ is a continuous function on H that satisfies the following averaging property*

$$\mathbb{E}\Big(\widetilde{Q}_g(x + M_r)\Big) = g(x) \tag{2.3}$$

for every $x > \widehat{a}$. We assume further that there exists $\widehat{x} \in H$ such that $\widetilde{Q}_g(\widehat{x}) = 0$, $\widetilde{Q}_g(x)$ is non-decreasing for $x > \widehat{x}$ and $\widetilde{Q}_g(x) \leq 0$ for $\widehat{a} < x < \widehat{x}$. Then the value function to the optimal stopping problem (1.1) is given by the formula $V(x) = \mathbb{E}_x(e^{-r\tau^*}g(X_{\tau^*}))$. Here x^* is the largest root of $\widetilde{Q}_g(x) = 0$ in $(\widehat{a}, \infty)$ and $\tau^* = \inf\{t > 0 : X_t > x^*\}$. Moreover, we have for every $x \in \mathbb{R}$, $V(x) = \mathbb{E}\Big(\widetilde{Q}_g(x + M_r)1_{\{x+M_r>x^*\}}\Big)$.

Example 2.2 *(Perpetual American call option). Consider the function $g(x) = \sum_{m=1}^M h_m e^{\theta_m x}$. Simple algebra shows that the function*

$$\widetilde{Q}_g(x) = \sum_{m=1}^{M} h_m e^{\theta_m x}\Big(\psi_r^+(-\mathrm{i}\theta_m)\Big)^{-1} \tag{2.4}$$

satisfies the averaging property (2.3) for all x. (Here we assume that $\mathbb{E}(e^{\theta_m M_r}) < \infty$, $1 \leq m \leq M$.) In particular, if $g(x) = e^x - K$, then $\widetilde{Q}_g(x) = e^x\Big(\psi_r^+(-\mathrm{i})\Big)^{-1} - K$. Denote by x_c^* the unique value such that $e^{x_c^*} = \psi_r^+(-\mathrm{i})K$. Then we have $V(x) = \mathbb{E}_x(e^{-r\tau^*}g(X_{\tau^*})) = \mathbb{E}\Big(\widetilde{Q}_g(x + M_r)1_{\{x+M_r>x_c^*\}}\Big)$, where $\tau^* = \inf\{t > 0 : X_t > x_c^*\}$.

Theorem 2.3 *Given a reward function g with $H \equiv \{g > 0\} = (-\infty, \widehat{a})$ for some $\widehat{a} > -\infty$. Suppose that $\widetilde{P}_g$ is a continuous function on H that satisfies the following averaging property*

$$\mathbb{E}\Big(\widetilde{P}_g(x + I_r)\Big) = g(x) \tag{2.5}$$

for every $x < \widehat{a}$. We assume further that there exists $\widehat{x} \in H$ such that $\widetilde{P}_g(\widehat{x}) = 0$, $\widetilde{P}_g(x)$ is non-increasing for $x < \widehat{x}$ and $\widetilde{P}_g(x) \leq 0$ for $\widehat{x} < x < \widehat{a}$. Then the value function to the optimal stopping problem (1.1) is given by the formula $V(x) = \mathbb{E}_x(e^{-r\tau^*} g(X_{\tau^*}))$, where x^* is the smallest root of $\widetilde{P}_g(x) = 0$ in $(-\infty, \widehat{a})$ and $\tau^* = \inf\{t > 0 : X_t < x^*\}$. Moreover, we have for every $x \in \mathbb{R}$, $V(x) = \mathbb{E}\bigg(\widetilde{P}_g(x + I_r)1_{\{x+I_r<x^*\}}\bigg)$.

Proof. We first write $\widehat{g}(x) = g(-x)$ and $\widehat{H} := \{\widehat{g} > 0\} = (-\widehat{a}, \infty)$. Set $\widetilde{Q}_{\widehat{g}}(x) = \widetilde{P}_g(-x)$ for $x > -\widehat{a}$ and $\widehat{M}_r := \sup_{0 \leq t \leq e_r} -X_t$. Observe that $\widetilde{Q}_{\widehat{g}}$ is a continuous function on $\widehat{H}$ that satisfies the averaging property (2.3) for $\widehat{M}_r$ and $\widehat{g}$ on $\widehat{H}$. Also, by assumption, there exists $-\widehat{x} \in \widehat{H}$ such that $\widetilde{Q}_{\widehat{g}}(-\widehat{x}) = 0$, $\widetilde{Q}_{\widehat{g}}(x)$ is nondecreasing for $x > -\widehat{x}$ and $\widetilde{Q}_{\widehat{g}}(x) \leq 0$ for $-\widehat{a} < x < -\widehat{x}$. Set $Y_t = -X_t$. By Theorem 2.1, we observe $\widehat{W}(y) := \sup_\tau \mathbb{E}_y[e^{-r\tau}\widehat{g}(Y_\tau)] = \mathbb{E}_y[e^{-r\tau^*}\widehat{g}(Y_{\tau^*})]$, where $\tau^* = \inf\{t > 0 : -X_t > -x^*\}$ and $-x^*$ is the largest root of $\widetilde{Q}_{\widehat{g}}(x) = 0$ in $(-\widehat{a}, \infty)$. Also, note that $V(x) = \sup_\tau \mathbb{E}_x[e^{-r\tau}g(X_\tau)] = \sup_\tau \mathbb{E}[e^{-r\tau}g(x+X_\tau)] = \sup_\tau \mathbb{E}[e^{-r\tau}\widehat{g}(-x-X_\tau)] = \widehat{W}(-x)$. Therefore, if we set $y = -x$ and $y^* = -x^*$ then we have

$$\begin{aligned} V(x) = \widehat{W}(y) =& \mathbb{E}_y\left[e^{-r\tau^*}\widehat{g}(Y_{\tau^*})\right] = \mathbb{E}\left[e^{-r\tau^*}\widehat{g}(y + Y_{\tau^*})\right] = \mathbb{E}\left[e^{-r\tau^*}g(-y - Y_{\tau^*})\right] \\ =& \mathbb{E}\left[e^{-r\tau^*}g(x + X_{\tau^*})\right] = \mathbb{E}_x\left[e^{-r\tau^*}g(X_{\tau^*})\right] \end{aligned}$$

and

$$\begin{aligned} V(x) = \widehat{W}(y) =& \mathbb{E}\left[\widetilde{Q}_{\widehat{g}}(y + \widehat{M}_r)1_{\{y+\widehat{M}_r>y^*\}}\right] = \mathbb{E}\left[\widetilde{P}_{\widehat{g}}(-y - \widehat{M}_r)1_{\{-y-\widehat{M}_r<-y^*\}}\right] \\ =& \mathbb{E}\left[\widetilde{P}_{\widehat{g}}(-y + I_r)1_{\{-y+I_r<-y^*\}}\right] = \mathbb{E}\left[\widetilde{P}_{\widehat{g}}(x + I_r)1_{\{x+I_r<x^*\}}\right], \end{aligned}$$

as required. The proof is complete.

Remark 2.4 *Given a reward function g with $\{g > 0\} = (-\infty, \widehat{a})$. Set $\widehat{g}(y) = g(-y)$ and $\widehat{M}_r := \sup_{0 \leq t \leq e_r} -X_t$ and assume that $\widetilde{Q}_{\widehat{g}}$ satisfies the averaging property (2.3) for $\widehat{M}_r$ and $\widehat{g}$(i.e., $\widehat{g}(y) = \mathbb{E}\left[\widetilde{Q}_{\widehat{g}}(y+\widehat{M}_r)\right]$ for all $y > -\widehat{a}$). Write $\widetilde{P}_g(x) = \widetilde{Q}_{\widehat{g}}(-x)$ for $x < \widehat{a}$. Then we have*

$$g(x) = \widehat{g}(-x) = \mathbb{E}\left[\widetilde{Q}_{\widehat{g}}(-x + \widehat{M}_r)\right] = \mathbb{E}\left[\widetilde{Q}_{\widehat{g}}(-x - I_r)\right] = \mathbb{E}\left[\widetilde{P}_g(x + I_r)\right], x < \widehat{a}. \tag{2.6}$$

Therefore, $\widetilde{P}_g(x)$ satisfies the averaging property (2.5) for I_r and g on $x < \widehat{a}$.

Example 2.5 *(Perpetual American put option). Consider the function $g(x) = \sum_{m=1}^{M} h_m e^{\theta_m x}$. Then the function*

$$\widetilde{P}_g(x) = \sum_{m=1}^{M} h_m e^{\theta_m x}\bigg(\psi_r^-(-\mathrm{i}\theta_m)\bigg)^{-1} \tag{2.7}$$

satisfies the averaging property (2.5). (Here we assume that $\mathbb{E}(e^{\theta_m I_r}) < \infty$, $1 \leq m \leq M$.) In particular, if $g(x) = K - e^x$, then $\widetilde{P}_g(x) = K - e^x\bigg(\psi_r^-(-\mathrm{i})\bigg)^{-1}$. Denote by x_p^* the unique value such that $e^{x_p^*} = \psi_r^-(-\mathrm{i})K$. Then we have $V(x) = \mathbb{E}_x(e^{-r\tau^*}g(X_{\tau^*})) = \mathbb{E}\bigg(\widetilde{P}_g(x+I_r)1_{\{x+I_r<x_p^*\}}\bigg)$, where $\tau^* = \inf\{t > 0 : X_t < x_p^*\}$.

From now on, we consider the jump-diffusion process X of the form

$$X_t = X_0 + at + bW_t + \sum_{n=1}^{N_t^\lambda} Y_n - \sum_{k=1}^{N_t^\mu} Z_k, \qquad t \geq 0. \tag{2.8}$$

Here, $a \in \mathbb{R}\backslash\{0\}$, $b \geq 0$, $W = (W_t, t \geq 0)$ is a standard Brownian motion, $N^\lambda = (N_t^\lambda; t \geq 0)$ and $N^\mu = (N_t^\mu; t \geq 0)$ are Poisson processes with rate $\lambda > 0$ and $\mu > 0$, respectively. Also, $Y = (Y_n, n \in \mathbb{N})$ and $Z = (Z_k, k \in \mathbb{N})$ are sequences of independent random variables with the identical matrix-exponential distribution given by

$$dF^{(+)}(x) = p_1(x)dx = 1_{\{x>0\}} \sum_{k=1}^{v_1} \sum_{j=1}^{n_k} \frac{c_{kj} \beta_k^j x^{j-1}}{(j-1)!} e^{-\beta_k x} dx$$

and

$$dF^{(-)}(x) = p_2(x)dx = 1_{\{x>0\}} \sum_{p=1}^{v_2} \sum_{m=1}^{\ell_p} \frac{\widetilde{c}_{pm} \alpha_p^m x^{m-1}}{(m-1)!} e^{-\alpha_p x} dx,$$

respectively. Here, the parameters $c_{kj}, \beta_k, \widetilde{c}_{pm}$, and α_p can in principle take complex values, but if we order α_p and β_k by their real parts then α_1 and β_1 must be real, while the others may be complex with $0 < \beta_1 < \mathcal{R}e\,(\beta_2) \leq \cdots \leq \mathcal{R}e\,(\beta_{v_1})$ and $0 < \alpha_1 < \mathcal{R}e\,(\alpha_2) \leq \cdots \leq \mathcal{R}e\,(\alpha_{v_2})$. The random variable W, N^λ, N^μ, Y and Z are assumed to be independent. Note that the characteristic exponent of X is given by

$$\psi(z) = \mathrm{i}az - \frac{b^2 z^2}{2} + \lambda \left[\sum_{k=1}^{v_1} \sum_{j=1}^{n_k} c_{kj} \left(\frac{\mathrm{i}\beta_k}{z + \mathrm{i}\beta_k} \right)^j - 1 \right] + \mu \left[\sum_{p=1}^{v_2} \sum_{m=1}^{\ell_p} \widetilde{c}_{pm} \left(\frac{-\mathrm{i}\alpha_p}{z - \mathrm{i}\alpha_p} \right)^m - 1 \right]. \tag{2.9}$$

Denote by $-\mathrm{i}\widetilde{\rho}_1, \cdots, -\mathrm{i}\widetilde{\rho}_{\mu_2}, -\mathrm{i}\rho_1, \cdots, -\mathrm{i}\rho_{\mu_1}$ the roots of $r - \psi(z) = 0$ with $\mathcal{R}e\,(\widetilde{\rho}_{\mu_2}) \leq \cdots \leq \mathcal{R}e\,(\widetilde{\rho}_1) \leq 0 < \mathcal{R}e\,(\rho_1) \leq \mathcal{R}e\,(\rho_2) \leq \cdots \leq \mathcal{R}e\,(\rho_{\mu_1})$. Note that $-\mathrm{i}\widetilde{\rho}_1$ and $-\mathrm{i}\rho_1$ are purely imaginary. Moreover, if $v_1 \geq 1$, then $0 < \rho_1 < \beta_1$ and if $v_2 \geq 1$, then $0 < -\widetilde{\rho}_1 < \alpha_1$. We assume further that all roots are simple. We observe that the distribution of I_r has the form

$$f_{I_r}(y)dy = 1_{\{a>0,b=0\}} \widetilde{d}_0 \delta_0(dy) + 1_{\{\mu_2 \geq 1\}} \left[\sum_{\eta=1}^{\mu_2} \widetilde{d}_\eta \widetilde{\rho}_\eta e^{-\widetilde{\rho}_\eta y} 1_{\{y<0\}} dy \right] \tag{2.10}$$

where $\widetilde{d}_0 = \prod_{j=1}^{\mu_2} (-\widetilde{\rho}_j) \prod_{k=1}^{v_2} \alpha_k^{-\ell_k}$ and

$$\widetilde{d}_j = -\prod_{k=1}^{v_2} \left(\frac{\widetilde{\rho}_j + \alpha_k}{\alpha_k} \right)^{\ell_k} \prod_{m=1, m \neq j}^{\mu_2} \frac{\widetilde{\rho}_m}{-\widetilde{\rho}_j + \widetilde{\rho}_m}, \text{ for } 1 \leq j \leq \mu_2. \tag{2.11}$$

Also the distribution of M_r is given by the formula

$$f_{M_r}(y)dy = 1_{\{a<0,b=0\}} d_0 \delta_0(dy) + 1_{\{\mu_1 \geq 1\}} \left[\sum_{j=1}^{\mu_1} d_j \rho_j e^{-\rho_j y} 1_{\{y>0\}} dy \right] \tag{2.12}$$

where $d_0 = \prod_{j=1}^{\mu_1} \rho_j \prod_{k=1}^{v_1} \beta_k^{-n_k}$ and

$$d_k = \prod_{j=1}^{v_1} \left(\frac{\beta_j - \rho_k}{\beta_j} \right)^{n_j} \prod_{i=1, i \neq k}^{\mu_1} \frac{\rho_i}{\rho_i - \rho_k}, \text{ for } 1 \leq k \leq \mu_1. \tag{2.13}$$

Definition 2.6 *We write $g \in \pi_0$ if the function $g : \mathbb{R} \to \mathbb{R}$ is continuously differentiable and for $v_1 \geq 1$, there exist $A_1 > 0, A_2 > 0$ and $\theta \in (0, \beta_1)$ such that $|g(x)| \leq A_1 + A_2 e^{\theta x}, \forall x \geq 0$. We write $g \in \pi_1$ if the function $g : \mathbb{R} \to \mathbb{R}$ is continuously differentiable and for $v_2 \geq 1$, there exist $A_1 > 0, A_2 > 0$ and $\theta \in (0, -\alpha_1)$ such that $|g(x)| \leq A_1 + A_2 e^{-\theta x}, \forall x \leq 0$.*

For any $g \in \pi_0$, we define the function $Q_g(x)$ by the formula

$$\begin{aligned} Q_g(x) \quad = \quad & 1_{\{\mu_2 \geq 1\}} \sum_{\eta=1}^{\mu_2} \frac{\widetilde{d}_\eta \widetilde{\rho}_\eta}{r} \left\{ \sum_{k=1}^{v_1} \sum_{j=1}^{n_k} \sum_{\ell=1}^{j} \frac{-\lambda(\beta_k)^j c_{kj}}{(\beta_k - \widetilde{\rho}_\eta)^\ell (j-\ell)!} e^{\beta_k x} \int_x^\infty (y-x)^{j-\ell} g(y) e^{-\beta_k y} dy \right. \\ & \left. - \left((a + \frac{b^2 \widetilde{\rho}_\eta}{2}) g(x) + \frac{b^2}{2} g'(x) \right) \right\} \\ & + 1_{\{a>0, b=0\}} \frac{\widetilde{d}_0}{r} \left\{ \sum_{k=1}^{v_1} \sum_{j=1}^{n_k} \frac{-\lambda(\beta_k)^j c_{kj}}{(j-1)!} e^{\beta_k x} \int_x^\infty (u-x)^{j-1} g(u) e^{-\beta_k u} du \right. \\ & \left. + (\lambda + \mu + r) g(x) - a g'(x) \right\}. \end{aligned} \tag{2.14}$$

On the other hand, if $g \in \pi_1$, the function P_g is define by the formula

$$\begin{aligned} P_g(x) = & \sum_{p=1}^{v_2} \sum_{m=1}^{\ell_p} \sum_{k=1}^{m} \frac{-\mu(\alpha_p)^m \widetilde{c}_{pm}}{r(m-k)!} \\ & \times \left[\left(1_{\{\mu_1 \geq 1\}} \sum_{j=1}^{\mu_1} \frac{d_j \rho_j}{(\alpha_p + \rho_j)^k} + 1_{\{a<0, b=0\}} 1_{\{k=1\}} d_0 \right) \int_{-\infty}^0 (-t)^{m-k} g(t+x) e^{\alpha_p t} dt \right] \\ & + \left[1_{\{\mu_1 \geq 1\}} \frac{b^2}{2r} \left(\sum_{j=1}^{\mu_1} d_j \rho_j \right) - 1_{\{a<0, b=0\}} \frac{a d_0}{r} \right] g'(x) \\ & + \left[1_{\{\mu_1 \geq 1\}} \sum_{j=1}^{\mu_1} \frac{d_j \rho_j}{r} (a + \frac{b^2 \rho_j}{2}) + 1_{\{a<0, b=0\}} \frac{d_0}{r} (\lambda + \mu + r) \right] g(x). \end{aligned} \tag{2.15}$$

Remark 2.7 *(a) Given two reward functions g and $\widehat{g}$. If $g = \widehat{g}$ on $[x, \infty)$, then $Q_g(x) = Q_{\widehat{g}}(x)$. If $g = \widehat{g}$ on $(-\infty, x]$, then $P_g(x) = P_{\widehat{g}}(x)$. (b) Consider the function $g(x) = \sum_{m=1}^M h_m e^{\theta_m x}$. If $\theta_m < \beta_1$ for all m, then we have $Q_g(x) = \sum_{m=1}^M h_m e^{\theta_m x} \left(\psi_r^+(-\mathrm{i}\theta_m) \right)^{-1}$. On the other hand, if $-\theta_m < \alpha_1$ for all m, then we have $P_g(x) = \sum_{m=1}^M h_m e^{\theta_m x} \left(\psi_r^-(-\mathrm{i}\theta_m) \right)^{-1}$.*

Proposition 2.8 *Given a reward function g. If $g \in \pi_0$, then the function $Q_g(x)$ given in (2.14) satisfies the averaging property (2.3) for all x. On the other hand, if $g \in \pi_1$, then the function $P_g(x)$ given in (2.15) satisfies the averaging property (2.5) for all x.*

Proof. The first statement was proved in Sheu and Tsai [15]. The second statement follows by using the fact that $P_g(-x) = Q_{\widehat{g}}(x)$ where $Q_{\widehat{g}}$ is given in (2.14) for $\widehat{g}(x) = g(-x)$ and the process $-X_t$.

3 American compound option

In this section, we consider the pricing problem of the perpetual American compound options. The perpetual American compound option have two strikes prices. For example, the call-on-call option gives its holder the right to buy at an random time τ for the strike price K_1 a call option with the strike price K_2 and the exercise time ζ, where $\zeta \geq \tau$. The compound option will be exercised on the first random time τ only the value of the option on that date is greater than the first strike price. The rational prices of perpetual American options can be formulated by the values of the optimal stopping problems

$$\text{(call-on-call)} \qquad V_1(x) = \sup_{\tau} \mathbb{E}_x\Big[e^{-r\tau} H_1^+(X_\tau)\Big]. \tag{3.16}$$

$$\text{(call-on-put)} \qquad V_2(x) = \sup_{\tau} \mathbb{E}_x\Big[e^{-r\tau} H_2^+(X_\tau)\Big]. \tag{3.17}$$

$$\text{(put-on-call)} \qquad V_3(x) = \sup_{\tau} \mathbb{E}_x\Big[e^{-r\tau} H_3^+(X_\tau)\Big]. \tag{3.18}$$

$$\text{(put-on-put)} \qquad V_4(x) = \sup_{\tau} \mathbb{E}_x\Big[e^{-r\tau} H_4^+(X_\tau)\Big]. \tag{3.19}$$

Here the reward functions $H_j(x)$, $j = 1, ..., 4$, are given by

$$H_1(x) = W(x) - K_1, \quad H_2(x) = U(x) - K_1, \quad H_3(x) = L_1 - W(x), \quad H_4(x) = L_1 - U(x) \tag{3.20}$$

for all $x \in \mathbb{R}$. Also, $W(x)$ and $U(x)$ denote the rational prices of the perpetual American call and put options with the strike prices K_2 and L_2, respectively and are given by

$$W(x) = \sup_{\eta} \mathbb{E}_x\Big[e^{-r\eta}(e^{X_\eta} - K_2)^+\Big] \text{ and } U(x) = \sup_{\eta} \mathbb{E}_x\Big[e^{-r\eta}(L_2 - e^{X_\eta})^+\Big] \tag{3.21}$$

where the suprema are taken over the stopping times η of the process X.

From now on, we assume that $\{X_t\}_{t\geq 0}$ is the form in (2.8) with $n_k = 1$, $\ell_p = 1$, $c_{k1} > 0$, $\beta_k > 0$, $\widetilde{c}_{p1} > 0$ and $\alpha_p > 0$, for $1 \leq k \leq v_1$ and $1 \leq p \leq v_2$. For simplicity, we assume that $b \neq 0$. In this case, $\mu_1 = v_1 + 1$, $\mu_2 = v_2 + 1$ and all roots are are simple and purely imaginary. Also they satisfy the conditions

$$0 < \rho_1 < \beta_1 < \rho_2 < \cdots < \beta_{\mu_1 - 1} < \rho_{\mu_1} \tag{3.22}$$

and

$$0 < -\widetilde{\rho}_1 < \alpha_1 < -\widetilde{\rho}_2 < \cdots < \alpha_{\mu_2 - 1} < -\widetilde{\rho}_{\mu_2}. \tag{3.23}$$

We assume further that $\rho_1 > 1$ and $-\widetilde{\rho}_1 > 1$. Recall that $f_{M_r}(y) = \sum_{j=1}^{\mu_1} d_j \rho_j e^{-\rho_j y} 1_{\{y>0\}}$ and $f_{I_r}(y) = \sum_{\eta=1}^{\mu_2} \widetilde{d}_\eta \widetilde{\rho}_\eta e^{-\widetilde{\rho}_\eta y} 1_{\{y<0\}}$, where

$$d_k = \prod_{j=1}^{v_1} \frac{\beta_j - \rho_k}{\beta_j} \prod_{i=1, i\neq k}^{\mu_1} \frac{\rho_i}{\rho_i - \rho_k}, \text{ for } 1 \leq k \leq \mu_1. \tag{3.24}$$

and

$$\widetilde{d}_\eta = -\prod_{k=1}^{v_2} \frac{\widetilde{\rho}_\eta + \alpha_k}{\alpha_k} \prod_{m=1, m\neq\eta}^{\mu_2} \frac{\widetilde{\rho}_m}{-\widetilde{\rho}_\eta + \widetilde{\rho}_m}, \text{ for } 1 \leq \eta \leq \mu_2 \tag{3.25}$$

From these, we observe $\psi_r^+(u) = \sum_{k=1}^{\mu_1} \frac{d_k \rho_k \mathrm{i}}{u + \mathrm{i}\rho_k}$ and $\psi_r^-(u) = -\sum_{\eta=1}^{\mu_2} \frac{\widetilde{d}_\eta \widetilde{\rho}_\eta \mathrm{i}}{u + \mathrm{i}\widetilde{\rho}_\eta}$. Also if H is in π_0, we have

$$\begin{aligned} Q_H(x) = &-\sum_{k=1}^{v_1}\sum_{\eta=1}^{\mu_2} \frac{\widetilde{d}_\eta \widetilde{\rho}_\eta \lambda \beta_k c_{k1}}{r(\beta_k - \widetilde{\rho}_\eta)} e^{\beta_k x} \int_x^\infty H(y) e^{-\beta_k y} dy \\ &- \sum_{\eta=1}^{\mu_2} \frac{\widetilde{d}_\eta \widetilde{\rho}_\eta}{r} (a + \frac{b^2 \widetilde{\rho}_\eta}{2}) H(x) - \sum_{\eta=1}^{\mu_2} \frac{\widetilde{d}_\eta \widetilde{\rho}_\eta b^2 H'(x)}{2r}. \end{aligned} \tag{3.26}$$

If H is in π_1, we have

$$\begin{aligned} P_H(x) = &-\sum_{p=1}^{v_2}\sum_{j=1}^{\mu_1} \frac{d_j \rho_j \mu \alpha_p \widetilde{c}_{p1}}{r(\alpha + \rho_j)} \int_{-\infty}^0 H(t+x) e^{\alpha_p t} dt \\ &+ \sum_{j=1}^{\mu_1} \frac{d_j \rho_j}{r} (a + \frac{b^2 \rho_j}{2}) H(x) + \sum_{j=1}^{\mu_1} \frac{d_j \rho_j b^2 H'(x)}{2r}. \end{aligned} \tag{3.27}$$

Set $g_1(x) = e^x - K_2$. Then $Q_{g_1}(x) = e^x \Big(\psi_r^+(-\mathrm{i})\Big)^{-1} - K_2$. Denote by x_c^* the unique value such that $e^{x_c^*} = \psi_r^+(-\mathrm{i}) K_2$. Then the function $W(x)$ in (3.21) is given by the formula

$$W(x) = 1_{\{x \geq x_c^*\}}(e^x - K_2) + 1_{\{x < x_c^*\}} \sum_{j=1}^{\mu_1} \frac{d_j K_2 e^{\rho_j (x - x_c^*)}}{\rho_j - 1}. \tag{3.28}$$

By (3.22) and (3.24), we obtain that $d_k > 0$ for all k. Hence, $W(x)$ is a strictly increasing function with $\lim_{x\to-\infty} W(x) = 0$ and $\lim_{x\to\infty} W(x) = \infty$. On the other hand, set $g_2(x) = L_2 - e^x$. Then $P_{g_2}(x) = L_2 - \Big(\psi_r^-(-\mathrm{i})\Big)^{-1} e^x$. Denote by x_p^* the unique value such that $e^{x_p^*} = \psi_r^-(-\mathrm{i}) L_2$. The value function $U(x)$ in (3.21)is given by the formula

$$U(x) = 1_{\{x \leq x_p^*\}}(L_2 - e^x) + 1_{\{x > x_p^*\}} \sum_{\eta=1}^{\mu_2} \frac{-\widetilde{d}_\eta L_2 e^{\widetilde{\rho}_\eta (x - x_p^*)}}{1 - \widetilde{\rho}_\eta}. \tag{3.29}$$

By (3.23) and (3.25), $\widetilde{d}_\eta < 0$ for all η and so $U(x)$ is a strictly decreasing function with $\lim_{x\to-\infty} U(x) = L_2$ and $\lim_{x\to\infty} W(x) = 0$.

(**Call-on-Call Option**). We consider the call-on-call option. The reward function $H_1(x)$ is given by

$$H_1(x) = W(x) - K_1 = 1_{\{x \geq x_c^*\}}(e^x - K_1 - K_2) + 1_{\{x < x_c^*\}} \Bigg(\sum_{j=1}^{\mu_1} \frac{d_j K_2 e^{\rho_j (x - x_c^*)}}{\rho_j - 1} - K_1 \Bigg). \tag{3.30}$$

Clearly, $H_1(x)$ is a strictly increasing function with $\lim_{x\to-\infty} H_1(x) = -K_1$ and $\lim_{x\to\infty} H_1(x) = \infty$. Hence there exists a unique $\widehat{a}_1 > -\infty$ such that $\{H_1 > 0\} = (\widehat{a}_1, \infty)$. Note that $H_1 \in \pi_0$ and $Q_{H_1}(x) = e^x \Big(\psi_r^+(-\mathrm{i})\Big)^{-1} - K_1 - K_2$ for $x \geq x_c^*$. Furthermore, if there exists x_1^* such that $Q_{H_1}(x_1^*) = 0, Q_{H_1}(x) \le 0$ for $\widehat{a}_1 < x < x_1^*$ and $Q_{H_1}(x)$ is non-decreasing on (x_1^*, ∞) then by Theorem 2.1, we deduce that $V_1(x) = \mathbb{E}_x(e^{-r\tau_1^*} H_1(X_{\tau_1^*})) = \int_{x_1^*-x}^{\infty} Q_{H_1}(x+m) f_{M_r}(m)dm$, where $\tau_1^* = \inf\{t > 0 : X_t > x_1^*\}$.

(**Call-on-Put Option**). We consider the call-on-put option. Then we have

$$H_2(x) = U(x) - K_1 = 1_{\{x \le x_p^*\}}(L_2 - K_1 - e^x) + 1_{\{x > x_p^*\}}\left(\sum_{\eta=1}^{\mu_2} \frac{-\widetilde{d}_\eta L_2 e^{\widetilde{\rho}_\eta (x - x_p^*)}}{1 - \widetilde{\rho}_\eta} - K_1\right). \quad (3.31)$$

Clearly, $H_2(x)$ is a strictly decreasing function with $\lim_{x\to-\infty} H_2(x) = L_2 - K_1$ and $\lim_{x\to\infty} H_2(x) = -K_1$. Hence $\{H_2 > 0\} = (-\infty, \widehat{a}_2)$ for some $\widehat{a}_2 < \infty$(for this compound option, we always assume that $L_2 > K_1$). Notice that $H_2 \in \pi_1$ and $P_{H_2}(x) = L_2 - K_1 - e^x \left(\psi_r^-(-\mathrm{i})\right)^{-1}$. for all $x \le x_p^*$. Furthermore, if there exists x_2^* such that $P_{H_2}(x_2^*) = 0, P_{H_2}(x) \le 0$ for $x_2^* < x < \widehat{a}_2$ and $P_{H_2}(x)$ is non-increasing on $(-\infty, x_2^*)$ then by Theorem 2.3, we conclude that $V_2(x) = \mathbb{E}_x(e^{-r\tau_2^*} H_2(X_{\tau_2^*})) = \int_{-\infty}^{x_2^*-x} P_{H_2}(x+y) f_{I_r}(y)dy$, where $\tau_2^* = \inf\{t > 0 : X_t < x_2^*\}$.

(**Put-on-Call Option**). We consider the put-on-call option. The payoff function is given by

$$H_3(x) = L_1 - W(x) = 1_{\{x \ge x_c^*\}}(L_1 + K_2 - e^x) + 1_{\{x < x_c^*\}}\left(L_1 - \sum_{j=1}^{\mu_1} \frac{d_j K_2 e^{\rho_j (x - x_c^*)}}{\rho_j - 1}\right). \quad (3.32)$$

Clearly, $H_3(x)$ is a strictly decreasing function with $\lim_{x\to-\infty} H_3(x) = L_1$ and $\lim_{x\to\infty} H_3(x) = -\infty$. Hence $\{H_3 > 0\} = (-\infty, \widehat{a}_3)$ for some $\widehat{a}_3 < \infty$. Notice that $H_3 \in \pi_1$ and $P_{H_3}(x) = L_1 - \sum_{j=1}^{\mu_1} \frac{d_j K_2 e^{\rho_j (x - x_c^*)}}{\rho_j - 1} \left(\psi_r^-(-\mathrm{i}\rho_j)\right)^{-1}$ for all $x \le x_c^*$. Furthermore, if there exists x_3^* such that $P_{H_2}(x_3^*) = 0, P_{H_3}(x) \le 0$ for $x_3^* < x < \widehat{a}_3$ and $P_{H_3}(x)$ is non-increasing on $(-\infty, x_3^*)$ then by Theorem 2.3, we conclude that $V_3(x) = \mathbb{E}_x(e^{-r\tau_3^*} H_3(X_{\tau_3^*})) = \int_{-\infty}^{x_3^*-x} P_{H_3}(x+y) f_{I_r}(y)dy$, where $\tau_3^* = \inf\{t > 0 : X_t < x_3^*\}$.

(**Put-on-Put Option**). We consider the put-on-put compound option. Then we have

$$H_4(x) = L_1 - U(x) = 1_{\{x \le x_p^*\}}(e^x + L_1 - L_2) + 1_{\{x > x_p^*\}}\left(L_1 - \sum_{\eta=1}^{\mu_2} \frac{-\widetilde{d}_\eta L_2 e^{\widetilde{\rho}_\eta (x - x_p^*)}}{1 - \widetilde{\rho}_\eta}\right). \quad (3.33)$$

Clearly, $H_4(x)$ is a strictly increasing function with $\lim_{x\to-\infty} H_4(x) = L_1 - L_2$ and $\lim_{x\to\infty} H_4(x) = L_1$. Hence $\{H_4 > 0\} = (\widehat{a}_4, \infty)$ for some $\widehat{a}_4 > -\infty$(for this option, we assume that $L_1 < L_2$). Note that $H_4 \in \pi_0$ and $Q_{H_4}(x) = L_1 - \sum_{\eta=1}^{\mu_2} \frac{-\widetilde{d}_\eta L_2 e^{\widetilde{\rho}_\eta (x - x_p^*)}}{1 - \widetilde{\rho}_\eta} \left(\psi_r^+(-\mathrm{i}\widetilde{\rho}_\eta)\right)^{-1}$ for all $x \ge x_p^*$. Furthermore, if there exists x_4^* such that $Q_{H_4}(x_4^*) = 0, Q_{H_4}(x) \le 0$ for $\widehat{a}_4 < x < x_p^*$ and $Q_{H_4}(x)$ is non-decreasing on (x_4^*, ∞) then by Theorem 2.1, we deduce that $V_4(x) = \mathbb{E}_x(e^{-r\tau_4^*} H_4(X_{\tau_4^*})) = \int_{x_4^*-x}^{\infty} Q_{H_4}(x+m) f_{M_r}(m)dm$, where $\tau_4^* = \inf\{t > 0 : X_t > x_4^*\}$.

4 Verification of optimality

Recall that the jump-diffusion process X is of the form in (2.8). To prove the optimality, we assume further that $\{Y_i^\beta : i = 1, 2, ...\}$ and $\{Z_j^\alpha : j = 1, 2, ...\}$ are sequences of independent exponentially distributed random variables with parameters β and α, respectively. First, recall that $g_1(x) = e^x - K_2$ and $W(x) = 1_{\{x \geq x_c^*\}}(e^x - K_2) + 1_{\{x < x_c^*\}}(\frac{d_1 K_2}{\rho_1 - 1} e^{\rho_1(x - x_c^*)} + \frac{d_2 K_2}{\rho_2 - 1} e^{\rho_2(x - x_c^*)})$, where x_c^* is the unique value satisfying $e^{x_c^*} = \psi_r^+(-\mathrm{i})K_2$. In addition, $g_2(x) = L_2 - e^x$ and $U(x) = 1_{\{x \leq x_p^*\}}(L_2 - e^x) + 1_{\{x > x_p^*\}} \sum_{\eta=1}^2 \frac{-\widetilde{d}_\eta L_2 e^{\widetilde{\rho}_\eta (x - x_p^*)}}{1 - \widetilde{\rho}_\eta}$, where x_p^* is the unique value satisfying $e^{x_p^*} = \psi_r^-(-\mathrm{i})L_2$.

(**Call-on-Call Option**). Note that $H_1(x) = W(x) - K_1 = 1_{\{x \geq x_c^*\}}(e^x - K_1 - K_2) + 1_{\{x \leq x_c^*\}}\Big(\frac{d_1 K_2}{\rho_1 - 1} e^{\rho_1(x - x_c^*)} + \frac{d_2 K_2}{\rho_2 - 1} e^{\rho_2(x - x_c^*)} - K_1\Big)$ and

$$
\begin{aligned}
Q_{H_1}(x) =& \Big(\frac{\widetilde{d}_1 \widetilde{\rho}_1}{r} \cdot \frac{-\lambda\beta}{\beta - \widetilde{\rho}_1} + \frac{\widetilde{d}_2 \widetilde{\rho}_2}{r} \cdot \frac{-\lambda\beta}{\beta - \widetilde{\rho}_2}\Big) e^{\beta x} \int_x^\infty H_1(y) e^{-\beta y} dy \\
& - \Big[\frac{\widetilde{d}_1 \widetilde{\rho}_1}{r} \cdot (a + \frac{b^2 \widetilde{\rho}_1}{2}) + \frac{\widetilde{d}_2 \widetilde{\rho}_2}{r} \cdot (a + \frac{b^2 \widetilde{\rho}_2}{2})\Big] H_1(x) - \Big[\frac{b^2}{2r}(\widetilde{d}_1 \widetilde{\rho}_1 + \widetilde{d}_2 \widetilde{\rho}_2)\Big] H_1^{'}(x).
\end{aligned} \tag{4.34}
$$

We show that the rational price of the call-on-call option is the rational price of the perpetual American call option with the strike price $K_1 + K_2$. That is $V_1(x) = \mathbb{E}_x(e^{-r\tau_1^*} H_1(X_{\tau_1^*})) = 1_{\{x \geq x_1^*\}}(e^x - K_1 - K_2) + 1_{\{x < x_1^*\}}(\frac{d_1(K_1 + K_2)}{\rho_1 - 1} e^{\rho_1(x - x_1^*)} + \frac{d_2(K_1 + K_2)}{\rho_2 - 1} e^{\rho_2(x - x_1^*)})$. Here x_1^* is the unique solution of $e^x(\psi_r^+(-\mathrm{i}))^{-1} - K_1 - K_2 = 0$ and $\tau_1^* = \inf\{t > 0 : X_t > x_1^*\}$. (Note that $x_1^* > x_c^*$.)

Case 1: $e^{x_c^*} - K_1 - K_2 \leq 0$. Our result follows from the fact that $H_1^+(x) = (e^x - K_1 - K_2)^+$.

Case 2: $e^{x_c^*} - K_1 - K_2 > 0$. In this case we have $\{H_1 > 0\} = (\widehat{a}_1, \infty)$ for some $\widehat{a}_1 < x_c^*$. For $\widehat{a}_1 < x < x_c^*$, we have

$$
\begin{aligned}
e^{\beta x} \int_x^\infty H_1(y) e^{-\beta y} dy =& e^{\beta(x - x_c^*)} \Big(\frac{d_1 K_2}{(\rho_1 - 1)(\rho_1 - \beta)} + \frac{d_2 K_2}{(\rho_2 - 1)(\rho_2 - \beta)} + \frac{\psi_r^+(-\mathrm{i})K_2}{\beta - 1} - \frac{K_2}{\beta}\Big) \\
& - \frac{d_1 K_2 e^{\rho_1(x - x_c^*)}}{(\rho_1 - 1)(\rho_1 - \beta)} - \frac{d_2 K_2 e^{\rho_2(x - x_c^*)}}{(\rho_2 - 1)(\rho_2 - \beta)} - \frac{K_1}{\beta}.
\end{aligned}
$$

Plugging this into (4.34) gives

$$
\begin{aligned}
Q_{H_1}(x) =& \Big(\frac{\widetilde{d}_1 \widetilde{\rho}_1}{r} \cdot \frac{-\lambda\beta}{\beta - \widetilde{\rho}_1} + \frac{\widetilde{d}_2 \widetilde{\rho}_2}{r} \cdot \frac{-\lambda\beta}{\beta - \widetilde{\rho}_2}\Big) \\
& \times \Big[e^{\beta(x - x_c^*)} \Big(\frac{d_1 K_2}{(\rho_1 - 1)(\rho_1 - \beta)} + \frac{d_2 K_2}{(\rho_2 - 1)(\rho_2 - \beta)} + \frac{\psi_r^+(-\mathrm{i})K_2}{\beta - 1} - \frac{K_2}{\beta}\Big) \\
& - \frac{d_1 K_2 e^{\rho_1(x - x_c^*)}}{(\rho_1 - 1)(\rho_1 - \beta)} - \frac{d_2 K_2 e^{\rho_2(x - x_c^*)}}{(\rho_2 - 1)(\rho_2 - \beta)} - \frac{K_1}{\beta}\Big] \\
& - \Big[\frac{\widetilde{d}_1 \widetilde{\rho}_1}{r} \cdot (a + \frac{b^2 \widetilde{\rho}_1}{2}) + \frac{\widetilde{d}_2 \widetilde{\rho}_2}{r} \cdot (a + \frac{b^2 \widetilde{\rho}_2}{2})\Big] \Big((\frac{d_1 K_2}{\rho_1 - 1} e^{\rho_1(x - x_c^*)} + \frac{d_2 K_2}{\rho_2 - 1} e^{\rho_2(x - x_c^*)}) - K_1\Big) \\
& - \Big[\frac{b^2}{2r}(\widetilde{d}_1 \widetilde{\rho}_1 + \widetilde{d}_2 \widetilde{\rho}_2)\Big] \Big(\frac{d_1 \rho_1 K_2}{\rho_1 - 1} e^{\rho_1(x - x_c^*)} + \frac{d_2 \rho_2 K_2}{\rho_2 - 1} e^{\rho_2(x - x_c^*)}\Big).
\end{aligned} \tag{4.35}
$$

Notice that

$$\begin{aligned}&\frac{d_1K_2}{(\rho_1-1)(\rho_1-\beta)}+\frac{d_2K_2}{(\rho_2-1)(\rho_2-\beta)}+\frac{\psi_r^+(-\mathrm{i})K_2}{\beta-1}-\frac{K_2}{\beta}\\=&\frac{d_1K_2}{(\rho_1-1)(\rho_1-\beta)}+\frac{d_2K_2}{(\rho_2-1)(\rho_2-\beta)}+(\frac{K_2}{\beta-1})(\frac{d_1\rho_1}{\rho_1-1}+\frac{d_2\rho_2}{\rho_2-1})-\frac{K_2}{\beta}\\=&\frac{K_2}{\beta(\beta-1)}\left(\frac{d_1\rho_1}{\rho_1-\beta}+\frac{d_2\rho_2}{\rho_2-\beta}\right)=0.\end{aligned}\tag{4.36}$$

Also, using the Wiener-Hopf factorization formula, we have that

$$\begin{aligned}\frac{r}{r-\psi(-\mathrm{i}z)}=&\frac{r(\beta-z)(\alpha+z)}{-\left[\frac{b^2z^2}{2}(\beta-z)(\alpha+z)+az(\beta-z)(\alpha+z)+\lambda z(\alpha+z)-\mu z(\beta-z)\right]+r(\beta-z)(\alpha+z)}\\=&\frac{\rho_1\rho_2(\beta-z)}{\beta(\rho_1-z)(\rho_2-z)}\sum_{k=1}^{2}\frac{\widetilde{d}_k\widetilde{\rho}_k}{z-\widetilde{\rho}_k}.\end{aligned}\tag{4.37}$$

Evaluating both sides of (4.37) at $z=\beta$ gives

$$\sum_{k=1}^{2}\frac{\widetilde{d}_k\widetilde{\rho}_k}{\beta-\widetilde{\rho}_k}=\frac{r(\rho_1-\beta)(\rho_2-\beta)}{-\lambda\rho_1\rho_2}.\tag{4.38}$$

Also, by multiplying both sides of (4.37) by z^2 and letting $z\to\infty$, we see that

$$\sum_{k=1}^{2}\widetilde{d}_k\widetilde{\rho}_k=\frac{r\beta}{\frac{b^2}{2}\rho_1\rho_2}.\tag{4.39}$$

Moreover, it follows from (a) and (c) in Lemma 3.1 of [15] that

$$\frac{\widetilde{d}_1\widetilde{\rho}_1}{r}\cdot(a+\frac{b^2\widetilde{\rho}_1}{2})+\frac{\widetilde{d}_2\widetilde{\rho}_2}{r}\cdot(a+\frac{b^2\widetilde{\rho}_2}{2})=\frac{1}{r}\bigg(-r-\frac{\lambda\widetilde{d}_1\widetilde{\rho}_1}{\beta-\widetilde{\rho}_1}-\frac{\lambda\widetilde{d}_2\widetilde{\rho}_2}{\beta-\widetilde{\rho}_2}\bigg).\tag{4.40}$$

Taking account (4.35)-(4.40) we have that for $\widehat{a}_1<x<x_c^*$,

$$\begin{aligned}Q_{H_1}(x)=&e^{\rho_1(x-x_c^*)}\frac{d_1K_2}{\rho_1-1}\left[-\frac{\beta(\rho_2-\beta)}{\rho_1\rho_2}+1-\frac{(\beta-\rho_1)(\beta-\rho_2)}{\rho_1\rho_2}-\frac{\beta}{\rho_2}\right]\\&+e^{\rho_2(x-x_c^*)}\frac{d_2K_2}{\rho_2-1}\left[-\frac{\beta(\rho_1-\beta)}{\rho_1\rho_2}+1-\frac{(\beta-\rho_1)(\beta-\rho_2)}{\rho_1\rho_2}-\frac{\beta}{\rho_1}\right]-K_1\\=&-K_1.\end{aligned}\tag{4.41}$$

For $x>x_c^*$, we have $H_1(x)=e^x-K_1-K_2$ and, hence, $Q_{H_1}(x)=Q_{\widetilde{H}_1}(x)$, where $\widetilde{H}_1(x)=e^x-K_1-K_2$. By (2.4), $Q_{\widetilde{H}_1}(x)=e^x(\psi_r^+(-\mathrm{i}))^{-1}-K_1-K_2$. Denote by x_1^* the unique solution of $e^x(\psi_r^+(-\mathrm{i}))^{-1}-K_1-K_2=0$. Clearly, we have $x_1^*>x_c^*$, $Q_{H_1}(x_1^*)=Q_{\widetilde{H}_1}(x_1^*)=0$, $Q_{H_1}(x)=Q_{\widetilde{H}_1}(x)<0$ on (x_c^*,x_1^*) and $Q_{H_1}(x)=Q_{\widetilde{H}_1}(x)$ is increasing on (x_1^*,∞). By Theorem 2.1, $V_1(x)=\mathbb{E}_x(e^{-r\tau_1^*}H_1(X_{\tau_1^*}))=\int_{x_1^*-x}^{\infty}Q_{H_1}(x+m)f_{M_r}(m)dm=\int_{x_1^*-x}^{\infty}Q_{\widetilde{H}_1}(x+m)f_{M_r}(m)dm=1_{\{x\geq x_1^*\}}(e^x-K_1-K_2)+1_{\{x<x_1^*\}}(\frac{d_1(K_1+K_2)}{\rho_1-1}e^{\rho_1(x-x_1^*)}+\frac{d_2(K_1+K_2)}{\rho_2-1}e^{\rho_2(x-x_1^*)})$. This completes the proof. ■

(**Call-on-Put option**). Note that $H_2(x)=U(x)-K_1=1_{\{x\leq x_p^*\}}(L_2-K_1-e^x)+1_{\{x>x_p^*\}}\bigg(\sum_{\eta=1}^{2}\frac{-\widetilde{d}_\eta L_2e^{\widetilde{\rho}_\eta(x-x_p^*)}}{1-\widetilde{\rho}_\eta}-$

$K_1\Big)$ and

$$P_{H_2}(x) = \Big(\frac{d_1\rho_1}{r}\cdot\frac{-\mu\alpha}{\alpha+\rho_1} + \frac{d_2\rho_2}{r}\cdot\frac{-\mu\alpha}{\alpha+\rho_2}\Big)e^{-\alpha x}\int_{-\infty}^{x} H_2(y)e^{\alpha y}dy$$
$$+\Big[\frac{d_1\rho_1}{r}\cdot(a+\frac{b^2\rho_1}{2}) + \frac{d_2\rho_2}{r}\cdot(a+\frac{b^2\rho_2}{2})\Big]H_2(x) + \Big[\frac{b^2}{2r}(d_1\rho_1+d_2\rho_2)\Big]H_2^{'}(x). \quad (4.42)$$

We show that the rational price of the call-on-put option is the rational price of the perpetual American put option with the strike price $L_2 - K_1$. That is $V_2(x) = \mathbb{E}_x(e^{-r\tau_2^*}H_2(X_{\tau_2^*})) = 1_{\{x\le x_2^*\}}(L_2 - K_1 - e^x) + 1_{\{x>x_2^*\}}\sum_{\eta=1}^{2}\frac{-\widetilde{d}_\eta(L_2-K_1)e^{\widetilde{\rho}_\eta(x-x_2^*)}}{1-\widetilde{\rho}_\eta}$. Here x_2^* is the unique solution of $L_2 - K_1 - e^x(\psi_r^-(-\mathrm{i}))^{-1} = 0$ and $\tau_2^* = \inf\{t>0 : X_t < x_2^*\}$. (Note that $x_2^* < x_p^*$.)

Case 1: $L_2 - K_1 \le e^{x_p^*}$. Our result follows from the fact that $H_2^+(x) = (L_2 - K_1 - e^x)^+$.

Case 2: $L_2 - K_1 > e^{x_p^*}$. In this case, we observe $\{H_2 > 0\} = (-\infty, \widehat{a}_2)$ for some $\widehat{a}_2 > x_p^*$. For $x_p^* < x < \widehat{a}_2$, we first observe that

$$e^{-\alpha x}\int_{-\infty}^{x} H_2(u)e^{\alpha u}du = e^{-\alpha(x-x_p^*)}\Big[\frac{\widetilde{d}_1L_2}{(\alpha+\widetilde{\rho}_1)(1-\widetilde{\rho}_1)} + \frac{\widetilde{d}_2L_2}{(\alpha+\widetilde{\rho}_2)(1-\widetilde{\rho}_2)} + \frac{L_2}{\alpha} - \frac{e^{x_p^*}}{\alpha+1}\Big]$$
$$+ e^{-\widetilde{\rho}_1(x_p^*-x)}\frac{-\widetilde{d}_1L_2}{(\alpha+\widetilde{\rho}_1)(1-\widetilde{\rho}_1)} + e^{-\widetilde{\rho}_2(x_p^*-x)}\frac{-\widetilde{d}_2L_2}{(\alpha+\widetilde{\rho}_2)(1-\widetilde{\rho}_2)} - \frac{K_1}{\alpha} \quad (4.43)$$

Also, since $L_2 - \Big(\psi_r^-(-\mathrm{i})\Big)^{-1}e^{x_p^*} = 0$ and $\psi_r^-(-\mathrm{i}) = \frac{\widetilde{d}_1\widetilde{\rho}_1}{1-\widetilde{\rho}_1} + \frac{\widetilde{d}_2\widetilde{\rho}_2}{1-\widetilde{\rho}_2}$, we have

$$\frac{\widetilde{d}_1L_2}{(\alpha+\widetilde{\rho}_1)(1-\widetilde{\rho}_1)} + \frac{\widetilde{d}_2L_2}{(\alpha+\widetilde{\rho}_2)(1-\widetilde{\rho}_2)} + \frac{L_2}{\alpha} - \frac{e^{x_p^*}}{\alpha+1} = \frac{-L_2}{\alpha(\alpha+1)}\Big(\frac{\widetilde{d}_1\widetilde{\rho}_1}{\alpha+\widetilde{\rho}_1} + \frac{\widetilde{d}_2\widetilde{\rho}_2}{\alpha+\widetilde{\rho}_2}\Big) = 0.$$

This together with (4.42) and (4.43) yields that for $x_p^* < x < \widehat{a}_2$,

$$P_{H_2}(x) = \Big(\frac{d_1\rho_1}{r}\cdot\frac{-\mu\alpha}{\alpha+\rho_1} + \frac{d_2\rho_2}{r}\cdot\frac{-\mu\alpha}{\alpha+\rho_2}\Big)\Big[\frac{-\widetilde{d}_1L_2e^{-\widetilde{\rho}_1(x_p^*-x)}}{(\alpha+\widetilde{\rho}_1)(1-\widetilde{\rho}_1)} + \frac{-\widetilde{d}_2L_2e^{-\widetilde{\rho}_2(x_p^*-x)}}{(\alpha+\widetilde{\rho}_2)(1-\widetilde{\rho}_2)} + \frac{-K_1}{\alpha}\Big]$$
$$+\Big[\frac{d_1\rho_1}{r}\cdot(a+\frac{b^2\rho_1}{2}) + \frac{d_2\rho_2}{r}\cdot(a+\frac{b^2\rho_2}{2})\Big]\Big[\frac{-\widetilde{d}_1L_2e^{-\widetilde{\rho}_1(x_p^*-x)}}{1-\widetilde{\rho}_1} + \frac{-\widetilde{d}_2L_2e^{-\widetilde{\rho}_2(x_p^*-x)}}{1-\widetilde{\rho}_2} - K_1\Big]$$
$$+\Big[\frac{b^2}{2r}(d_1\rho_1+d_2\rho_2)\Big]\Big[\frac{-\widetilde{d}_1\widetilde{\rho}_1L_2e^{-\widetilde{\rho}_1(x_p^*-x)}}{1-\widetilde{\rho}_1} + \frac{-\widetilde{d}_2\widetilde{\rho}_2L_2e^{-\widetilde{\rho}_2(x_p^*-x)}}{1-\widetilde{\rho}_2}\Big]. \quad (4.44)$$

In addition, applying similar arguments as in (4.38)-(4.40), we obtain that

$$\frac{d_1\rho_1}{\alpha+\rho_1} + \frac{d_2\rho_2}{\alpha+\rho_2} = \frac{r(\widetilde{\rho}_1+\alpha)(\widetilde{\rho}_2+\alpha)}{-\mu\widetilde{\rho}_1\widetilde{\rho}_2} \quad (4.45)$$

$$\sum_{j=1}^{2} d_j\rho_j = \frac{r\alpha}{\frac{b^2}{2}\widetilde{\rho}_1\widetilde{\rho}_2} \quad (4.46)$$

and

$$d_1(a\rho_1 + \frac{b^2\rho_1^2}{2}) + d_2(a\rho_2 + \frac{b^2\rho_2^2}{2}) = r + \mu\Big(\frac{d_1\rho_1}{\alpha+\rho_1} + \frac{d_2\rho_2}{\alpha+\rho_2}\Big) \quad (4.47)$$

Plugging (4.45)-(4.47) into (4.44), we obtain that for $x_p^* < x < \widehat{a}_2$

$$\begin{aligned} P_{H_2}(x) =& \frac{-\widetilde{d}_1 L_2 e^{\widetilde{\rho}_1(x-x_p^*)}}{1-\widetilde{\rho}_1}\left[\frac{\alpha(\widetilde{\rho}_2+\alpha)}{\widetilde{\rho}_1\widetilde{\rho}_2}+1-\frac{(\widetilde{\rho}_1+\alpha)(\widetilde{\rho}_2+\alpha)}{\widetilde{\rho}_1\widetilde{\rho}_2}+\frac{\alpha}{\widetilde{\rho}_2}\right] \\ &+\frac{-\widetilde{d}_2 L_2 e^{\widetilde{\rho}_2(x-x_p^*)}}{1-\widetilde{\rho}_2}\left[\frac{\alpha(\widetilde{\rho}_1+\alpha)}{\widetilde{\rho}_1\widetilde{\rho}_2}+1-\frac{(\widetilde{\rho}_2+\alpha)(\widetilde{\rho}_1+\alpha)}{\widetilde{\rho}_1\widetilde{\rho}_2}+\frac{\alpha}{\widetilde{\rho}_1}\right]-K_1 \\ =& -K_1 \end{aligned} \tag{4.48}$$

Because $H_2(x) = L_2 - K_1 - e^x$ for $x < x_p^*$, we have $P_{H_2}(x) = P_{\widetilde{H}_2}(x)$ for $x < x_p^*$, where $\widetilde{H}_2 = L_2 - K_1 - e^x$. By (2.7), $P_{\widetilde{H}_2}(x) = L_2 - K_1 - e^x(\psi_r^-(-\mathrm{i}))^{-1}$. Denote by x_2^* the unique solution of $L_2 - K_1 - e^x(\psi_r^-(-\mathrm{i}))^{-1} = 0$. Then we have $x_2^* < x_p^*$. By Theorem 2.3,$V_2(x) = \mathbb{E}_x(e^{-r\tau_2^*}H_2(X_{\tau_2^*})) = \int_{-\infty}^{x_2^*-x} P_{H_2}(x+z)f_{I_r}(z)dz = \int_{-\infty}^{x_2^*-x} P_{\widetilde{H}_2}(x+z)f_{I_r}(z)dz = 1_{\{x\le x_2^*\}}(L_2 - K_1 - e^x) + 1_{\{x>x_2^*\}}\sum_{\eta=1}^{2}\frac{-\widetilde{d}_\eta(L_2-K_1)e^{\widetilde{\rho}_\eta(x-x_2^*)}}{1-\widetilde{\rho}_\eta}$. ■

(**Put-on-Call option**). Note that $H_3(x) = L_1 - W(x) = 1_{\{x\ge x_c^*\}}(L_1+K_2-e^x)+1_{\{x\le x_c^*\}}(L_1 - K_2(\frac{d_1}{\rho_1-1}e^{\rho_1(x-x_c^*)}+\frac{d_2}{\rho_2-1}e^{\rho_2(x-x_c^*)}))$ and

$$\begin{aligned} P_{H_3}(x) =& \left(\frac{d_1\rho_1}{r}\cdot\frac{-\mu\alpha}{\alpha+\rho_1}+\frac{d_2\rho_2}{r}\cdot\frac{-\mu\alpha}{\alpha+\rho_2}\right)e^{-\alpha x}\int_{-\infty}^{x}H_3(y)e^{\alpha y}dy \\ &+\left[\frac{d_1\rho_1}{r}\cdot(a+\frac{b^2\rho_1}{2})+\frac{d_2\rho_2}{r}\cdot(a+\frac{b^2\rho_2}{2})\right]H_3(x)+\left[\frac{b^2}{2r}(d_1\rho_1+d_2\rho_2)\right]H_3'(x). \end{aligned} \tag{4.49}$$

Case 1: $L_1 + K_2 - e^{x_c^*} \le 0$. In this case, we have $\{H_3 > 0\} = (-\infty, \widehat{a}_3)$ for some $\widehat{a}_3 < x_c^*$ and, hence, $P_{H_3}(x) = L_1 - \sum_{j=1}^{2}\frac{d_jK_2(\psi_r^-(-\mathrm{i}\rho_j))^{-1}e^{\rho_j(x-x_c^*)}}{\rho_j-1}$ for all $x < \widehat{a}_3$. Note that $P_{H_3}(x)$ is strictly decreasing with $\lim_{x\to-\infty}P_{H_3}(x) = L_1$ and $\lim_{x\to\widehat{a}_3}P_{H_3}(x) < 0$(???). Hence, there exists unique $x_3^* < \widehat{a}_3 < x_c^*$ such that $P_{H_3}(x_3^*) = 0$. By Theorem 2.3, we deduce that $V_3(x) = \mathbb{E}_x(e^{-r\tau_3^*}H_3(X_{\tau_3^*})) = \int_{-\infty}^{x_3^*-x}P_{H_3}(x+z)f_{I_r}(z)dz$, where $\tau_3^* = \inf\{t > 0 : X_t < x_3^*\}$.

Case 2: $L_1 + K_2 - e^{x_c^*} > 0$. In this case we have $\{H_3 > 0\} = (-\infty, \widehat{a}_3)$ for some $x_c^* < \widehat{a}_3$. For $x_c^* < x < \widehat{a}_3$, direct calculation gives

$$\begin{aligned} &e^{-\alpha x}\int_{-\infty}^{x}H_3(u)e^{\alpha u}du \\ =&e^{-\alpha(x-x_c^*)}\left(-\frac{d_1K_2}{(\rho_1-1)(\rho_1+\alpha)}-\frac{d_2K_2}{(\rho_2-1)(\rho_2+\alpha)}+\left(\frac{d_1\rho_1}{\rho_1-1}+\frac{d_2\rho_2}{\rho_2-1}\right)\frac{K_2}{\alpha+1}-\frac{K_2}{\alpha}\right)-\frac{e^x}{\alpha+1}+\frac{L_1+K_2}{\alpha} \\ =&\frac{K_2e^{-\alpha(x-x_c^*)}}{\alpha(\alpha+1)}(-\frac{d_1\rho_1}{\rho_1+\alpha}-\frac{d_2\rho_2}{\rho_2+\alpha})-\frac{e^x}{\alpha+1}+\frac{L_1+K_2}{\alpha}. \end{aligned} \tag{4.50}$$

In addition, by similar approach as in (4.38)-(4.40), we have that

$$\frac{d_1\rho_1}{\alpha+\rho_1}+\frac{d_2\rho_2}{\alpha+\rho_2}=\frac{r(\widetilde{\rho}_1+\alpha)(\widetilde{\rho}_2+\alpha)}{-\mu\widetilde{\rho}_1\widetilde{\rho}_2}, \tag{4.51}$$

$$\sum_{j=1}^{2}d_j\rho_j=\frac{r\alpha}{\frac{b^2}{2}\widetilde{\rho}_1\widetilde{\rho}_2}, \tag{4.52}$$

and

$$d_1(a\rho_1+\frac{b^2\rho_1^2}{2})+d_2(a\rho_2+\frac{b^2\rho_2^2}{2})=r+\mu\left(\frac{d_1\rho_1}{\alpha+\rho_1}+\frac{d_2\rho_2}{\alpha+\rho_2}\right). \tag{4.53}$$

Therefore, by (4.49) and (4.50)-(4.53), we have that for $x_c^* < x < \widehat{a}_3$,

$$\begin{aligned}
P_{H_3}(x) =& \Big(\frac{d_1\rho_1}{r} \cdot \frac{-\mu\alpha}{\alpha+\rho_1} + \frac{d_2\rho_2}{r} \cdot \frac{-\mu\alpha}{\alpha+\rho_2} \Big) \Big[\frac{K_2 e^{-\alpha(x-x_c^*)}}{\alpha(\alpha+1)} \Big(-\frac{d_1\rho_1}{\rho_1+\alpha} - \frac{d_2\rho_2}{\rho_2+\alpha} \Big) - \frac{e^x}{\alpha+1} + \frac{L_1+K_2}{\alpha} \Big] \\
&+ \Big[1 + \frac{\mu}{r} \Big(\frac{d_1\rho_1}{\alpha+\rho_1} + \frac{d_2\rho_2}{\alpha+\rho_2} \Big) \Big] (L_1 + K_2 - e^x) - \Big[\frac{b^2}{2r} (d_1\rho_1 + d_2\rho_2) \Big] e^x \\
=& \frac{\mu K_2}{r(\alpha+1)} \Big(\frac{d_1\rho_1}{\alpha+\rho_1} + \frac{d_2\rho_2}{\alpha+\rho_2} \Big)^2 e^{-\alpha(x-x_c^*)} \\
&+ \Big[-1 - \frac{\mu}{r(\alpha+1)} \Big(\frac{d_1\rho_1}{\alpha+\rho_1} + \frac{d_2\rho_2}{\alpha+\rho_2} \Big) - \frac{b^2}{2r}(d_1\rho_1 + d_2\rho_2) \Big] e^x + L_1 + K_2. \qquad (4.54)
\end{aligned}$$

From the identity above, we see that $P_{H_3}(x)$ is decreasing on $(x_c^*, \widehat{a}_3)$ and

$$\begin{aligned}
\lim_{x\to(x_c^*)^+} P_{H_3}(x) =& L_1 + K_2 + \frac{\mu K_2}{r(\alpha+1)} \Big(\frac{d_1\rho_1}{\alpha+\rho_1} + \frac{d_2\rho_2}{\alpha+\rho_2} \Big)^2 \\
&- \Big[1 + \frac{\mu}{r(\alpha+1)} \Big(\frac{d_1\rho_1}{\alpha+\rho_1} + \frac{d_2\rho_2}{\alpha+\rho_2} \Big) + \frac{b^2}{2r}(d_1\rho_1 + d_2\rho_2) \Big] e^{x_c^*}.
\end{aligned}$$

On $(-\infty, x_c^*)$, $P_{H_3}(x) = L_1 - \sum_{j=1}^2 \frac{d_j K_2 (\psi_r^-(-\mathrm{i}\rho_j))^{-1} e^{\rho_j(x-x_c^*)}}{\rho_j - 1}$ is decreasing on $(-\infty, x_c^*)$. Notice that $\lim_{x\to-\infty} P_{H_3}(x) = L_1$ and $\lim_{x\to\widehat{a}_3} P_{H_3}(x) < 0$(???). Hence, there is an unique $x_3^* < \widehat{a}_3$ such that $P_{H_3}(x_3^*) = 0$. (Note that if $\lim_{x\to(x_c^*)^+} P_{H_3}(x) < 0$, then $x_3^* < x_c^*$ and the optimal boundary and the rational price are identical to that for Case 1; otherwise, $x_3^* \geq x_c^*$.) By Theorem 2.3, we deduce that $V_3(x) = \mathbb{E}_x(e^{-r\tau_3^*} H_3(X_{\tau_3^*})) = \int_{-\infty}^{x_3^*-x} P_{H_3}(x+z) f_{I_r}(z) dz$, where $\tau_3^* = \inf\{t > 0 : X_t < x_3^*\}$. ■

(**Put-on-Put option**). Note that $H_4(x) = L_1 - U(x) = 1_{\{x \leq x_p^*\}}(e^x + L_1 - L_2) + 1_{\{x > x_p^*\}} \Big(L_1 - \sum_{\eta=1}^{\mu_2} \frac{-\widetilde{d}_\eta L_2 e^{\widetilde{\rho}_\eta (x-x_p^*)}}{1-\widetilde{\rho}_\eta} \Big)$ and

$$\begin{aligned}
Q_{H_4}(x) =& \Big(\frac{\widetilde{d}_1\widetilde{\rho}_1}{r} \cdot \frac{-\lambda\beta}{\beta-\widetilde{\rho}_1} + \frac{\widetilde{d}_2\widetilde{\rho}_2}{r} \cdot \frac{-\lambda\beta}{\beta-\widetilde{\rho}_2} \Big) e^{\beta x} \int_x^\infty H_4(y) e^{-\beta y} dy \\
&- \Big[\frac{\widetilde{d}_1\widetilde{\rho}_1}{r} \cdot (a + \frac{b^2\widetilde{\rho}_1}{2}) + \frac{\widetilde{d}_2\widetilde{\rho}_2}{r} \cdot (a + \frac{b^2\widetilde{\rho}_2}{2}) \Big] H_4(x) - \Big[\frac{b^2}{2r} (\widetilde{d}_1\widetilde{\rho}_1 + \widetilde{d}_2\widetilde{\rho}_2) \Big] H_4^{'}(x) \quad (4.55)
\end{aligned}$$

Case 1: $e^{x_p^*} + L_1 - L_2 \leq 0$. We have $\{H_4 > 0\} = (\widehat{a}_4, \infty)$ for some $\widehat{a}_4 > x_p^*$ and $Q_{H_4}(x) = L_1 - \sum_{\eta=1}^2 \frac{-\widetilde{d}_\eta L_2 (\psi_r^+(-\mathrm{i}\widetilde{\rho}_\eta))^{-1} e^{\widetilde{\rho}_\eta(x-x_p^*)}}{1-\widetilde{\rho}_\eta}$ for all $x > \widehat{a}_4$. Notice that $Q_{H_4}(x)$ is increasing with $\lim_{x\to\infty} Q_{H_4}(x) = L_1$ and $\lim_{x\to\widehat{a}_4} Q_{H_4}(x) < 0$(???). Hence, there exists an unique $x_4^* > \widehat{a}_4$ such that $Q_{H_4}(x_4^*) = 0$. By Theorem 2.1, we see that $V_4(x) = \mathbb{E}_x(e^{-r\tau_4^*} H_4(X_{\tau_4^*})) = \int_{x_4^*-x}^\infty Q_{H_4}(x + m) f_{M_r}(m) dm$, where $\tau_4^* = \inf\{t > 0 : X_t > x_4^*\}$.

Case 2: $e^{x_p^*} + L_1 - L_2 > 0$. For this case, we get $\{H_4 > 0\} = (\widehat{a}_4, \infty)$ for some $\widehat{a}_4 < x_p^*$. For

$\widehat{a}_4 < x < x_p^*$, we have

$$
\begin{aligned}
&e^{\beta x}\int_x^\infty H_4(y)e^{-\beta y}dy\\
=&e^{\beta(x-x_p^*)}\left(\frac{e^{x_p^*}}{1-\beta}+\frac{L_2}{\beta}+\frac{-\widetilde{d}_1L_2}{(1-\widetilde{\rho}_1)(\widetilde{\rho}_1-\beta)}+\frac{-\widetilde{d}_2L_2}{(1-\widetilde{\rho}_2)(\widetilde{\rho}_2-\beta)}\right)-\frac{e^x}{1-\beta}+\frac{L_1-L_2}{\beta}\\
=&\frac{L_2e^{\beta(x-x_p^*)}}{\beta(\beta-1)}\left(\frac{-\widetilde{d}_1\widetilde{\rho}_1}{\beta-\widetilde{\rho}_1}+\frac{-\widetilde{d}_2\widetilde{\rho}_2}{\beta-\widetilde{\rho}_2}\right)-\frac{e^x}{1-\beta}+\frac{L_1-L_2}{\beta}. \qquad (4.56)
\end{aligned}
$$

Plugging (4.56) into (4.55)and using (4.40) gives for $\widehat{a}_4 < x \leq x_p^*$,

$$
\begin{aligned}
Q_{H_4}(x)=&\left(\frac{\widetilde{d}_1\widetilde{\rho}_1}{r}\cdot\frac{-\lambda\beta}{\beta-\widetilde{\rho}_1}+\frac{\widetilde{d}_2\widetilde{\rho}_2}{r}\cdot\frac{-\lambda\beta}{\beta-\widetilde{\rho}_2}\right)\left[\frac{L_2e^{\beta(x-x_p^*)}}{\beta(\beta-1)}\left(\frac{-\widetilde{d}_1\widetilde{\rho}_1}{\beta-\widetilde{\rho}_1}+\frac{-\widetilde{d}_2\widetilde{\rho}_2}{\beta-\widetilde{\rho}_2}\right)-\frac{e^x}{1-\beta}+\frac{L_1-L_2}{\beta}\right]\\
&+\left[1+\frac{\lambda}{r}\left(\frac{\widetilde{d}_1\widetilde{\rho}_1}{\beta-\widetilde{\rho}_1}+\frac{\widetilde{d}_2\widetilde{\rho}_2}{\beta-\widetilde{\rho}_2}\right)\right]\left(e^x+L_1-L_2\right)-\left[\frac{b^2}{2r}(\widetilde{d}_1\widetilde{\rho}_1+\widetilde{d}_2\widetilde{\rho}_2)\right]e^x\\
=&\frac{\lambda L_2}{r(\beta-1)}\left(\frac{-\widetilde{d}_1\widetilde{\rho}_1}{\beta-\widetilde{\rho}_1}+\frac{-\widetilde{d}_2\widetilde{\rho}_2}{\beta-\widetilde{\rho}_2}\right)^2e^{\beta(x-x_p^*)}\\
&+\left[1+\frac{\lambda}{r(1-\beta)}\left(\frac{\widetilde{d}_1\widetilde{\rho}_1}{\beta-\widetilde{\rho}_1}+\frac{\widetilde{d}_2\widetilde{\rho}_2}{\beta-\widetilde{\rho}_2}\right)-\frac{b^2}{2r}(\widetilde{d}_1\widetilde{\rho}_1+\widetilde{d}_2\widetilde{\rho}_2)\right]e^x+L_1-L_2. \qquad (4.57)
\end{aligned}
$$

By using (4.38), (4.39) and the fact that $\beta\rho_1\rho_2 > \beta\rho_1+\beta\rho_2-\beta$, we obtain that

$$
1+\frac{\lambda}{r(1-\beta)}\left(\frac{\widetilde{d}_1\widetilde{\rho}_1}{\beta-\widetilde{\rho}_1}+\frac{\widetilde{d}_2\widetilde{\rho}_2}{\beta-\widetilde{\rho}_2}\right)-\frac{b^2}{2r}(\widetilde{d}_1\widetilde{\rho}_1+\widetilde{d}_2\widetilde{\rho}_2)=1-\frac{\beta\rho_1+\beta\rho_2-\beta-\rho_1\rho_2}{\rho_1\rho_2(\beta-1)}>0.
$$

This together with (4.57) leads to the facts that $Q_{H_4}(x)$ is increasing on $(\widehat{a}_4, x_p^*)$ and

$$
\begin{aligned}
\lim_{x\to(x_p^*)^-}Q_{H_4}(x)=&\frac{\lambda L_2}{r(\beta-1)}\left(\frac{-\widetilde{d}_1\widetilde{\rho}_1}{\beta-\widetilde{\rho}_1}+\frac{-\widetilde{d}_2\widetilde{\rho}_2}{\beta-\widetilde{\rho}_2}\right)^2\\
&+\left[1+\frac{\lambda}{r(1-\beta)}\left(\frac{\widetilde{d}_1\widetilde{\rho}_1}{\beta-\widetilde{\rho}_1}+\frac{\widetilde{d}_2\widetilde{\rho}_2}{\beta-\widetilde{\rho}_2}\right)-\frac{b^2}{2r}(\widetilde{d}_1\widetilde{\rho}_1+\widetilde{d}_2\widetilde{\rho}_2)\right]e^{x_p^*}+L_1-L_2
\end{aligned}
\qquad (4.58)
$$

As noted before,$Q_{H_4}(x)$ is increasing on (x_p^*, ∞). Also, notice that $\lim_{x\to\infty} Q_{H_4}(x) = L_1$ and $\lim_{x\to\widehat{a}_4} Q_{H_4}(x) < 0$. Hence, there exists an unique $x_4^* > \widehat{a}$ such that $Q_{H_4}(x_4^*) = 0$. (Note that if $\lim_{x\to(x_p^*)} Q_{H_4}(x) < 0$, then $x_4^* > x_p^*$ and the optimal boundary and the rational price are identical to that for Case 1; otherwise, $x_4^* \leq x_p^*$.) By Theorem 2.1, we see that $V_4(x) = \mathbb{E}_x(e^{-r\tau_4^*}H_4(X_{\tau_4^*})) = \int_{x_4^*-x}^\infty Q_{H_4}(x+m)f_{M_r}(m)dm$, where $\tau_4^* = \inf\{t > 0 : X_t > x_4^*\}$. ∎

Next, we consider the compound options for diffusion processes and assume that $X_t = at + bW_t$. In this case, $d_1 = 1, \widetilde{d}_1 = -1$ and $\widetilde{\rho}_1 < 0 < \rho_1$ are solutions of $ax + \frac{1}{2}b^2x^2 - r = 0$. Recall that $g_1(x) = e^x - K_2$ and $W(x) = 1_{\{x\geq x_c^*\}}(e^x - K_2) + 1_{\{x<x_c^*\}}\frac{d_1K_2}{\rho_1-1}e^{\rho_1(x-x_c^*)}$. Here x_c^* is the unique value satisfying $e^{x_c^*} = \frac{\rho_1K_2}{\rho_1-1}$. In addition, $g_2(x) = L_2 - e^x$ and $U(x) = 1_{\{x\leq x_p^*\}}(L_2 - e^x) + 1_{\{x>x_p^*\}}\frac{-\widetilde{d}_1L_2e^{\widetilde{\rho}_1(x-x_p^*)}}{1-\widetilde{\rho}_1}$, where x_p^* is the unique value such that $e^{x_p^*} = \frac{-\widetilde{\rho}_1L_2}{1-\widetilde{\rho}_1}$. (**Call-on-Call option**). Notice that $H_1(x) = 1_{\{x\geq x_c^*\}}(e^x-K_1-K_2)+1_{\{x\leq x_c^*\}}\Big(\frac{d_1K_2}{\rho_1-1}e^{\rho_1(x-x_c^*)} -$

$K_1\Big)$ and $Q_{H_1}(x) = -\frac{\widetilde{d}_1\widetilde{\rho}_1}{r}(a+\frac{b^2\widetilde{\rho}_1}{2})H_1(x) - \frac{b^2\widetilde{d}_1\widetilde{\rho}_1}{2r}H_1'(x)$. First, notice that if $e^{x_c^*} - K_1 - K_2 \le 0$, then $H_1^+(x) = (e^x - K_1 - K_2)^+$ and hence the rational price of the call-on-call option is the rational price of the perpetual American call option with the strike price K_1+K_2. Next, consider the case $e^{x_c^*} - K_1 - K_2 > 0$. Then $\{H_1 > 0\} = (\widehat{a}_1, \infty)$ for some $\widehat{a}_1 < x_c^*$. By using the facts that $d_1 = 1$, $\widetilde{d}_1 = -1$, $\frac{\widetilde{d}_1\widetilde{\rho}_1}{r}(a + \frac{b^2\widetilde{\rho}_1}{2}) = -1$ and $-\frac{b^2}{2r}\widetilde{\rho}_1\rho_1 = 1$, , we see that for $\widehat{a}_1 < x < x_c^*$

$$Q_{H_1}(x) = -\frac{\widetilde{d}_1\widetilde{\rho}_1}{r}\cdot(a+\frac{b^2\widetilde{\rho}_1}{2})\left(\frac{d_1K_2}{\rho_1-1}e^{\rho_1(x-x_c^*)} - K_1\right) - \frac{b^2}{2r}(\widetilde{d}_1\widetilde{\rho}_1)\left(\frac{d_1\rho_1K_2}{\rho_1-1}e^{\rho_1(x-x_c^*)}\right) = -K_1.$$

For $x \ge x_c^*$, we have $H_1(x) = e^x - K_1 - K_2$ and $Q_{H_1}(x) = e^x - K_1 - K_2 - \frac{e^x}{\rho_1}$. By the same argument as for the jump-diffusion processes, the rational price of the call-on-call option is the rational price of the perpetual American call option with the strike price $K_1 + K_2$. That is $V_1(x) = 1_{\{x \ge x_1^*\}}(e^x - K_1 - K_2) + 1_{\{x<x_1^*\}}\frac{d_1(K_1+K_2)}{\rho_1-1}e^{\rho_1(x-x_1^*)}$. Here x_1^* is the unique value satisfying $e^{x_1^*} = \frac{\rho_1(K_1+K_2)}{\rho_1-1}$. ■

(**Call-on-Put option**). Notice that $H_2(x) = 1_{\{x\le x_p^*\}}(L_2 - K_1 - e^x) + 1_{\{x>x_p^*\}}\Big(\frac{-\widetilde{d}_1L_2e^{\widetilde{\rho}_1(x-x_p^*)}}{1-\widetilde{\rho}_1} - K_1\Big)$ and $P_{H_2}(x) = \frac{d_1\rho_1}{r}(a + \frac{b^2\rho_1}{2})H_2(x) + \frac{b^2d_1\rho_1}{2r}H_2'(x)$. First, notice that if $L_2 - K_1 - e^{x_p^*} \le 0$ then $H_2^+(x) = (L_2 - K_1 - e^x)^+$ and the rational price of the call-on-put option is the rational price of the perpetual American put option with the strike price $L_2 - K_1$. Next, consider the case in which $L_2 - K_1 - e^{x_p^*} > 0$. Then $\{H_2 > 0\} = (-\infty, \widehat{a}_2)$ for some $\widehat{a}_2 > x_p^*$. Taking account of the facts that $d_1 = 1$, $\frac{d_1\rho_1}{r}(a + \frac{b^2\rho_1}{2}) = 1$ and $-\frac{d_1\rho_1\widetilde{\rho}_1}{2r} = 1$, we have that for $x_p^* < x < \widehat{a}_2, P_{H_2}(x) = \frac{L_2e^{\widetilde{\rho}_1(x-x_p^*)}}{1-\widetilde{\rho}_1} - K_1 + \frac{1}{(-\widetilde{\rho}_1)}\frac{\widetilde{\rho}_1L_2e^{\widetilde{\rho}_1(x-x_p^*)}}{1-\widetilde{\rho}_1} = -K_1$. Also, for $x \le x_p^*$, we have $H_2(x) = L_2 - K_1 - e^x$ and $P_{H_2}(x) = (-1 + \frac{1}{\rho_1})e^x + L_2 - K_1$. By the same argument as for the jump-diffusion processes, the rational price of the call-on-put option is the rational price of the perpetual American put option with the strike price $L_2 - K_1$. That is $V_2(x) = 1_{\{x\le x_2^*\}}(L_2 - K_1 - e^x) + 1_{\{x>x_2^*\}}\frac{-\widetilde{d}_1(L_2-K_1)e^{\widetilde{\rho}_1(x-x_2^*)}}{1-\widetilde{\rho}_1}$, where x_2^* is the unique value such that $e^{x_2^*} = \frac{-\widetilde{\rho}_1(L_2-K_1)}{1-\widetilde{\rho}_1}$. ■

(**Put-on-Call option**). Note that $H_3(x) = 1_{\{x\ge x_c^*\}}(L_1+K_2-e^x) + 1_{\{x\le x_c^*\}}(L_1 - \frac{d_1K_2}{\rho_1-1}e^{\rho_1(x-x_c^*)})$ and $P_{H_3}(x) = \frac{d_1\rho_1}{r}\cdot(a + \frac{b^2\rho_1}{2})H_3(x) + \frac{b^2}{2r}(d_1\rho_1)H_3'(x)$. First, notice that if $L_1 + K_2 \le e^{x_c^*}$, then $\{H_3 > 0\} = (-\infty, \widehat{a}_3)$ for some $\widehat{a}_3 < x_c^*$ and $H_3(x) = L_1 - \frac{d_1K_2}{\rho_1-1}e^{\rho_1(x-x_c^*)}$. It follows from (2.7) that $P_{H_3}(x) = L_1 - \frac{d_1K_2(\psi_r^-(-\mathrm{i}\rho_1))^{-1}e^{\rho_1(x-x_c^*)}}{\rho_1-1}$. Clearly $P_{H_3}(x)$ is strictly decreasing with $\lim_{x\to-\infty}P_{H_3}(x) = L_1$ and $\lim_{x\to\widehat{a}_3}P_{H_3}(x) < 0$(???). Therefore, there exists an unique $x_3^* < x_c^*$ such that $P_{H_3}(x_3^*) = 0$. By Theorem 2.3, we deduce that $V_3(x) = \int_{-\infty}^{x_3^*-x}P_{H_3}(x+z)f_{I_r}(z)dz$.

Next, consider the case $L_1 + K_2 > e^{x_c^*}$. We have $\{H_3 > 0\} = (-\infty, \widehat{a}_3)$ for some $\widehat{a}_3 > x_c^*$. Using the facts $d_1 = 1$, $\frac{d_1\rho_1}{r}(a + \frac{b^2\rho_1}{2}) = 1$ and $-\frac{d_1\rho_1\widetilde{\rho}_1b^2}{2r} = 1$, we get that for $x_c^* \le x < \widehat{a}_3$,

$$P_{H_3}(x) = \frac{d_1\rho_1}{r}(a + \frac{b^2\rho_1}{2})\Big[L_1 + K_2 - e^x\Big] - \frac{d_1\rho_1b^2}{2r}e^x = L_1 + K_2 + (\frac{1}{\widetilde{\rho}_1} - 1)e^x. \tag{4.59}$$

On the other hand, for $x < x_c^*, P_{H_3}(x) = L_1 - \frac{d_1K_2(\psi_r^-(-\mathrm{i}\rho_1))^{-1}e^{\rho_1(x-x_c^*)}}{\rho_1-1}$. Therefore $P_{H_3}(x)$ is a decreasing function on $(-\infty, \widehat{a}_3)$ and

$$\lim_{x\to(x_c^*)^+} P_{H_3}(x) = L_1 + K_2 + (\frac{1-\widetilde{\rho}_1}{\widetilde{\rho}_1})e^{x_c^*} = L_1 - \frac{\widetilde{\rho}_1 - \rho_1}{\widetilde{\rho}_1(\rho_1 - 1)}K_2. \tag{4.60}$$

If $L_1 < \frac{\widetilde{\rho}_1-\rho_1}{\widetilde{\rho}_1(\rho_1-1)}K_2$, then there is only one $x_3^* < x_c^*$ such that $P_{H_3}(x_3^*) = 0$, i.e., $e^{\rho_1 x_3^*} = \frac{L_1\widetilde{\rho}_1\rho_1 e^{(\rho_1-1)x_c^*}}{\widetilde{\rho}_1-\rho_1}$. Therefore, by Theorem 2.3, we deduce that x_3^* is the optimal boundary and for $x \geq x_3^*$,

$$\begin{aligned} V_3(x) = &\int_{-\infty}^{x_3^*} P_{H_3}(u) f_{I_r}(u-x)du = \int_{-\infty}^{x_3^*}\left[\frac{K_2 e^{\rho_1(u-x_c^*)}}{\rho_1 - 1}\left(-1+\frac{\rho_1}{\widetilde{\rho}_1}\right)+L_1\right](-\widetilde{\rho}_1)e^{-\widetilde{\rho}_1(u-x)}du \\ =&e^{\widetilde{\rho}_1(x-x_3^*)}\left(\frac{K_2 e^{-\rho_1(x_c^*-x_3^*)}}{1-\rho_1}+L_1\right) = -\frac{1}{\widetilde{\rho}_1}e^{x_c^*}\cdot e^{\widetilde{\rho}_1(x-x_3^*)}\cdot e^{\rho_1(x_3^*-x_c^*)} \end{aligned}$$

Also, for $x < x_3^*$, we have $V_3(x) = L_1 - \frac{d_1 K_2 e^{\rho_1(x-x_c^*)}}{\rho_1-1}$.

On the other hand, if $L_1 \geq \frac{\widetilde{\rho}_1-\rho_1}{\widetilde{\rho}_1(\rho_1-1)}K_2$, then there is only one $x_3^* \geq x_c^*$ such that $P_{H_3}(x_3^*) = 0$, that is $e^{x_3^*} = \frac{\widetilde{\rho}_1(L_1+K_2)}{\widetilde{\rho}_1-1}$. By Theorem 2.3, we deduce that x_3^* is the optimal boundary and the value function is given as follows. For $x > x_c^*$, we have

$$\begin{aligned} V_3(x) = &\int_{-\infty}^{x_c^*} P_H(u) f_{I_r}(u-x)du + \int_{x_c^*}^{x_3^*} P_H(u) f_{I_r}(u-x)du \\ =&e^{\widetilde{\rho}_1(x-x_c^*)}\left(e^{x_c^*}+\frac{\rho_1 K_2}{1-\rho_1}\right)+e^{\widetilde{\rho}_1(x-x_3^*)}\left(L_1+K_2-e^{x_{23}^*}\right) \\ =&e^{\widetilde{\rho}_1(x-x_3^*)}\frac{e^{x_3^*}}{-\widetilde{\rho}_1}. \end{aligned} \tag{4.61}$$

For $x < x_c^*, V_3(x) = L_1 - \frac{d_1 K_2 e^{\rho_1(x-x_c^*)}}{\rho_1-1}$ and for $x_c^* \leq x \leq x_3^*, V_3(x) = L_1 + K_2 - e^x$. ■

(**Put-on-Put option**). Notice that $H_4(x) = 1_{\{x\leq x_p^*\}}(e^x + L_1 - L_2) + 1_{\{x>x_p^*\}}\left(L_1 - \frac{-\widetilde{d}_1 L_2 e^{\widetilde{\rho}_1(x-x_p^*)}}{1-\widetilde{\rho}_1}\right)$ and $Q_{H_4}(x) = -\frac{\widetilde{d}_1\widetilde{\rho}_1}{r}\cdot(a+\frac{b^2\widetilde{\rho}_1}{2})H_4(x) - \frac{b^2\widetilde{d}_1\widetilde{\rho}_1}{2r}H_4'(x)$. First, notice that if $e^{x_p^*} + L_1 - L_2 \leq 0$, then $\{H_4 > 0\} = (\widehat{a}_4, \infty)$ for some $\widehat{a}_4 > x_4^*$ and $H_4(x) = L_1 - \frac{-\widetilde{d}_1 L_2 e^{\widetilde{\rho}_1(x-x_p^*)}}{1-\widetilde{\rho}_1}$. It follows from (2.4) that $Q_{H_4}(x) = \frac{L_2 e^{\widetilde{\rho}_1(x-x_p^*)}}{1-\widetilde{\rho}_1}(-1+\frac{\widetilde{\rho}_1}{\rho_1})+L_1$. Hence there exists an unique $x_4^* > x_p^*$ such that $Q_{H_4}(x_4^*) = 0$. By Theorem 2.1, x_4^* is the optimal boundary and $V_4 = \int_{x_4^*-x}^{\infty} Q_{H_4}(x+m) f_{M_r}(m)dm$, where $f_{M_r}(m) = d_1\rho_1 e^{-\rho_1 m}1_{\{m>0\}}$. Next, consider the case $e^{x_p^*} + L_1 - L_2 > 0$. Then we get $\{H_4 > 0\} = (\widehat{a}, \infty)$ for some $\widehat{a} < x_p^*$. By using the facts that $d_1 = 1$, $\widetilde{d}_1 = -1$, $\frac{\widetilde{d}_1\widetilde{\rho}_1}{r}\cdot(a+\frac{b^2\widetilde{\rho}_1}{2}) = -1$ and $-\frac{b^2}{2r}\widetilde{\rho}_1\rho_1 = 1$, we see that for $\widehat{a} < x \leq x_p^*$

$$Q_{H_4}(x) = e^x + L_1 - L_2 + \frac{e^x}{(-\rho_1)} = (1-\frac{1}{\rho_1})e^x + L_1 - L_2 \tag{4.62}$$

and for $x > x_p^*$

$$Q_{H_4}(x) = L_1 - \frac{L_2 e^{\widetilde{\rho}_1(x-x_p^*)}}{1-\widetilde{\rho}_1} + \frac{\widetilde{\rho}_1 L_2 e^{\widetilde{\rho}_1(x-x_p^*)}}{\rho_1(1-\widetilde{\rho}_1)} = \frac{L_2 e^{\widetilde{\rho}_1(x-x_p^*)}}{1-\widetilde{\rho}_1}(-1+\frac{\widetilde{\rho}_1}{\rho_1}) + L_1. \tag{4.63}$$

Therefore $Q_{H_4}(x)$ is an increasing function on $(\widehat{a}, \infty)$ with

$$\lim_{x\to(x_p^*)^-} Q_{H_4}(x) = (\frac{\rho_1 - 1}{\rho_1})e^{x_p^*} + L_1 - L_2 = \frac{(\widetilde{\rho}_1 - \rho_1)L_2}{\rho_1(1-\widetilde{\rho}_1)} + L_1. \tag{4.64}$$

If $\frac{(\widetilde{\rho}_1-\rho_1)L_2}{\rho_1(1-\widetilde{\rho}_1)} + L_1 < 0$ then there exists only one $x_4^* > x_p^*$ such that $Q_{H_4}(x_4^*) = 0$, that is $e^{\widetilde{\rho}_1 x_4^*} = \frac{L_1\rho_1\widetilde{\rho}_1 e^{\widetilde{\rho}_1 x_p^*}}{L_2(\widetilde{\rho}_1-\rho_1)e^{x_p^*}}$. By Theorem 2.1, we see that x_4^* is the optimal boundary and for $x < x_4^*$,

$$
\begin{aligned}
V_4(x) = \int_{x_4^*}^{\infty} Q_{H_4}(u) f_{M_r}(u-x)du = \int_{x_4^*}^{\infty} \left[\frac{L_2(\widetilde{\rho}_1-\rho_1)}{(1-\widetilde{\rho}_1)\rho_1} e^{\widetilde{\rho}_1(u-x_p^*)} + L_1\right]\rho_1 e^{-\rho_1(u-x)}du \\
= e^{\rho_1(x-x_4^*)} e^{\widetilde{\rho}_1(x_4^*-x_p^*)} \frac{-\widetilde{\rho}_1 L_2}{\rho_1(1-\widetilde{\rho}_1)} = e^{\rho_1(x-x_4^*)} e^{\widetilde{\rho}_1(x_4^*-x_p^*)} \frac{e^{x_p^*}}{\rho_1}
\end{aligned}
\tag{4.65}
$$

Also, for $x \geq x_4^*$, $V_4(x) = L_1 - \frac{(-\widetilde{d}_1)L_2 e^{\widetilde{\rho}_1(x-x_p^*)}}{1-\widetilde{\rho}_1}$. On the other hand, if $\frac{(\widetilde{\rho}_1-\rho_1)L_2}{\rho_1(1-\widetilde{\rho}_1)} + L_1 > 0$ then there is only one $x_4^* < x_p^*$ such that $Q_{H_4}(x_4^*) = 0$, i.e., $e^{x_4^*} = \frac{(L_2-L_1)\rho_1}{\rho_1-1}$. By Theorem 2.1, x_4^* is the optimal boundary and for $x < x_4^*$,

$$
\begin{aligned}
V_4(x) =& \int_{x_4^*}^{x_p^*} Q_{H_4}(u) f_{M_r}(u-x)du + \int_{x_p^*}^{\infty} Q_{H_4}(u) f_{M_r}(u-x)du \\
=& e^{\rho_1(x-x_p^*)}\left(e^{x_p^*} + L_2 - \frac{L_2}{1-\widetilde{\rho}_1}\right) + e^{\rho_1(x-x_4^*)}\left(e^{x_4^*} + L_1 - L_2\right) \\
=& e^{\rho_1(x-x_4^*)}\frac{e^{x_4^*}}{\rho_1}
\end{aligned}
\tag{4.66}
$$

Also, we have for $x > x_4^*$,

$$
V_4(x) = 1_{\{x_4^* \leq x \leq x_p^*\}} e^x - L_1 - L_2 + 1_{\{x > x_p^*\}} L_1 - \frac{(-\widetilde{d}_1)L_2 e^{\widetilde{\rho}_1(x-x_p^*)}}{1-\widetilde{\rho}_1}. \tag{4.67}
$$

5 Numerical Results

Example 5.1 *(call-on-call option). We consider the strike prices* $K_1 = 10$ *and* $K_2 = 50$. *For the diffusion process,* $x_c^* = 4.2120$, $x_1^* = 4.3943$, $(d_1, \widetilde{d}_1) = (1, -1)$ *and* $(\rho_1, \widetilde{\rho}_1) = (3.8577, -0.4977)$. *For the exponential jump-diffusion process, we acquire that* $x_c^* = 5.3363$, $x_1^* = 5.5186$, $(d_1, d_2, \widetilde{d}_1, \widetilde{d}_2) = (0.6275, 0.3724, -0.8955, -0.1044)$ *and* $(\rho_1, \rho_2, \widetilde{\rho}_1, \widetilde{\rho}_2) = (1.2029, 6.9435, -0.2359, -3.4791)$. *For the mixture-exponential jump-diffusion process, we have that* $x_c^* = 4.7666$, $x_1^* = 4.9490$, $(d_1, d_2, d_3, \widetilde{d}_1, \widetilde{d}_2, \widetilde{d}_3)$ $= (0.4405, 0.1445, 0.4149, -0.7975, -0.05012, -0.1523)$ *and* $(\rho_1, \rho_2, \rho_3, \widetilde{\rho}_1, \widetilde{\rho}_2, \widetilde{\rho}_3)$ $= (1.3605, 3.3113, 7.2730, -0.03213, -0.2879, -2.8148)$.

Example 5.2 *(call-on-put option). We consider the strike prices* $K_1 = 20$ *and* $L_2 = 50$. *For the diffusion process,* $x_p^* = 2.8103$, $x_2^* = 2.2995$, $(d_1, \widetilde{d}_1) = (1, -1)$ *and* $(\rho_1, \widetilde{\rho}_1) = (3.8577, -0.4977)$. *For the exponential jump-diffusion process, we acquire that* $x_p^* = 2.5340$, $x_2^* = 2.0232$, $(d_1, d_2, \widetilde{d}_1, \widetilde{d}_2) = (0.6275, 0.3724, -0.8955, -0.1044)$ *and* $(\rho_1, \rho_2, \widetilde{\rho}_1, \widetilde{\rho}_2) = (1.2029, 6.9435, -0.2359, -3.4791)$. *For the mixture-exponential jump-diffusion process, we have that* $x_p^* = 2.0042$, $x_2^* = 1.4934$, $(d_1, d_2, d_3, \widetilde{d}_1, \widetilde{d}_2, \widetilde{d}_3)$ $= (0.4405, 0.1445, 0.4149, -0.7975, -0.05012, -0.1523)$ *and* $(\rho_1, \rho_2, \rho_3, \widetilde{\rho}_1, \widetilde{\rho}_2, \widetilde{\rho}_3)$ $= (1.3605, 3.3113, 7.2730, -0.03213, -0.2879, -2.8148)$.

Example 5.3 *(put-on-call option). We consider the strike prices* $L_1 = 300$ *and* $K_2 = 50$. *For the diffusion process, we have that* $x_c^* = 4.2120$, $x_3^* = 4.7562$, $(d_1, \widetilde{d}_1) = (1, -1)$ *and*

$(\rho_1, \widetilde{\rho}_1) = (3.8577, -0.4977)$. *For the exponential jump-diffusion process, we acquire that* $x_c^* = 5.3363$, $x_3^* = 4.6442$, $(d_1, d_2, \widetilde{d}_1, \widetilde{d}_2) = (0.6275, 0.3724, -0.8955, -0.1044)$ *and* $(\rho_1, \rho_2, \widetilde{\rho}_1, \widetilde{\rho}_2) = (1.2029, 6.9435, -0.2359, -3.4791)$. *For the mixture-exponential jump-diffusion process, we have that* $x_c^* = 4.7666$, $x_3^* = 4.3958$, $(d_1, d_2, d_3, \widetilde{d}_1, \widetilde{d}_2, \widetilde{d}_3) = (0.4405, 0.1445, 0.4149, -0.7975, -0.0501, -0.1523)$ *and* $(\rho_1, \rho_2, \rho_3, \widetilde{\rho}_1, \widetilde{\rho}_2, \widetilde{\rho}_3) = (1.3605, 3.3113, 7.2730, -0.0321, -0.2879, -2.8148)$.

Example 5.4 *(put-on-put option). We consider the strike prices* $L_1 = 45$ *and* $L_2 = 50$. *For the diffusion process,* $x_p^* = 2.8103$, $x_4^* = 1.9094$, $(d_1, \widetilde{d}_1) = (1, -1)$ *and* $(\rho_1, \widetilde{\rho}_1) = (3.8577, -0.4977)$. *For the exponential jump-diffusion process, we acquire that* $x_p^* = 2.5340$, $x_4^* = 2.4091$, $(d_1, d_2, \widetilde{d}_1, \widetilde{d}_2) = (0.6275, 0.3724, -0.8955, -0.1044)$ *and* $(\rho_1, \rho_2, \widetilde{\rho}_1, \widetilde{\rho}_2) = (1.2029, 6.9435, -0.2359, -3.4791)$. *For the mixture-exponential jump-diffusion process, we acquire that* $x_p^* = 2.0042$, $x_4^* = 2.0176$, $(d_1, d_2, d_3, \widetilde{d}_1, \widetilde{d}_2, \widetilde{d}_3) = (0.4405, 0.1445, 0.4149, -0.7975, -0.0501, -0.1523)$ *and* $(\rho_1, \rho_2, \rho_3, \widetilde{\rho}_1, \widetilde{\rho}_2, \widetilde{\rho}_3) = (1.3605, 3.3113, 7.2730, -0.0321, -0.2879, -2.8148)$.

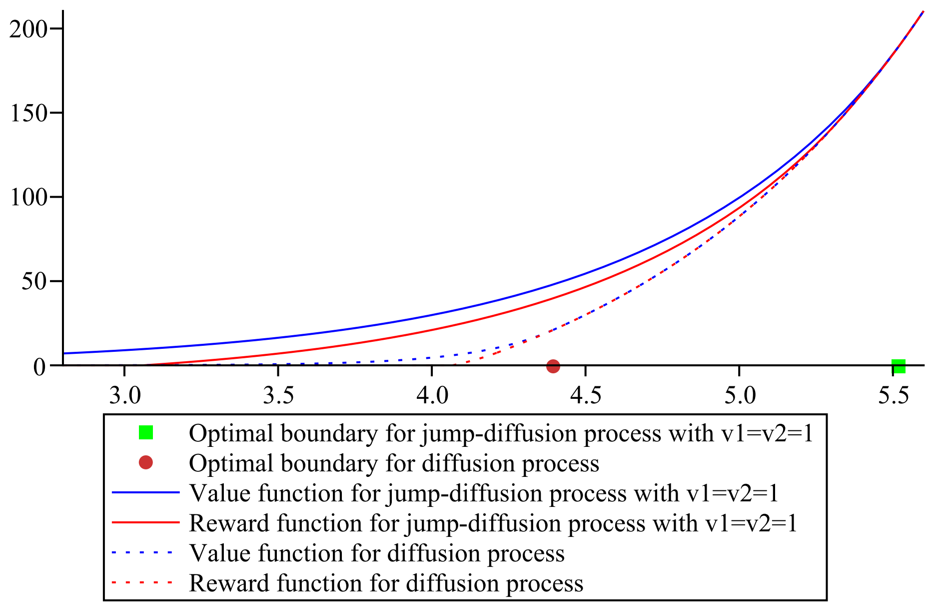

Figure 1: call-on-call options for jump-diffusion process with v1=v2=1 and diffusion process.

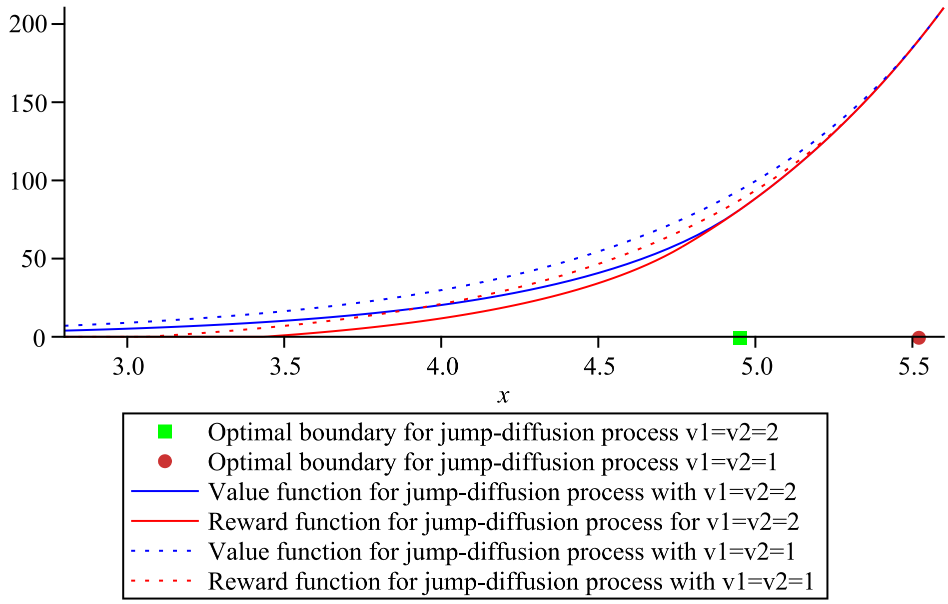

Figure 2: call-on-call options for jump-diffusion process with v1=v2=1 and v1=v2=2.

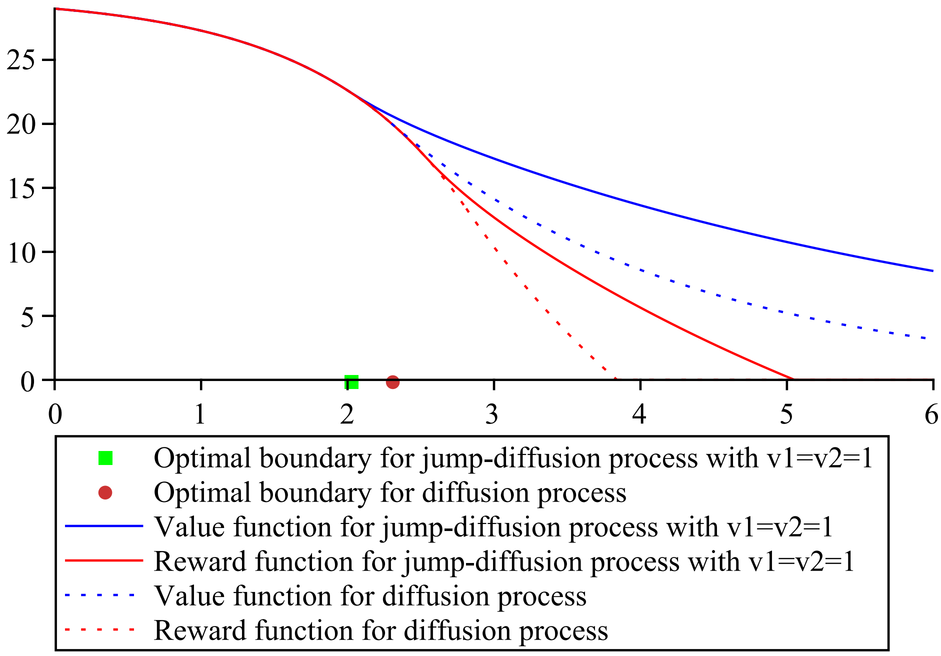

Figure 3: call-on-put options for jump-diffusion process with v1=v2=1 and diffusion process.

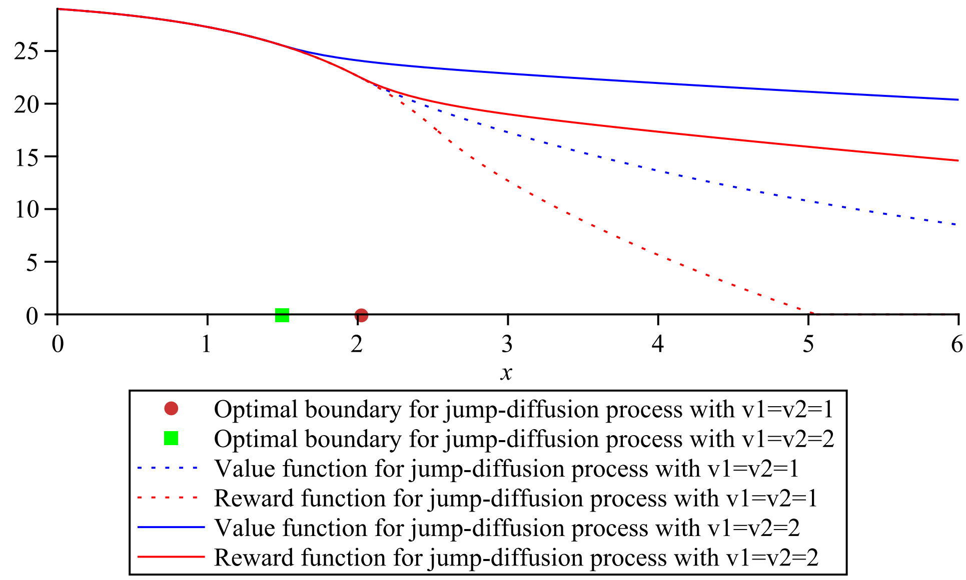

Figure 4: call-on-put options for jump-diffusion process with v1=v2=1 and v1=v2=2.

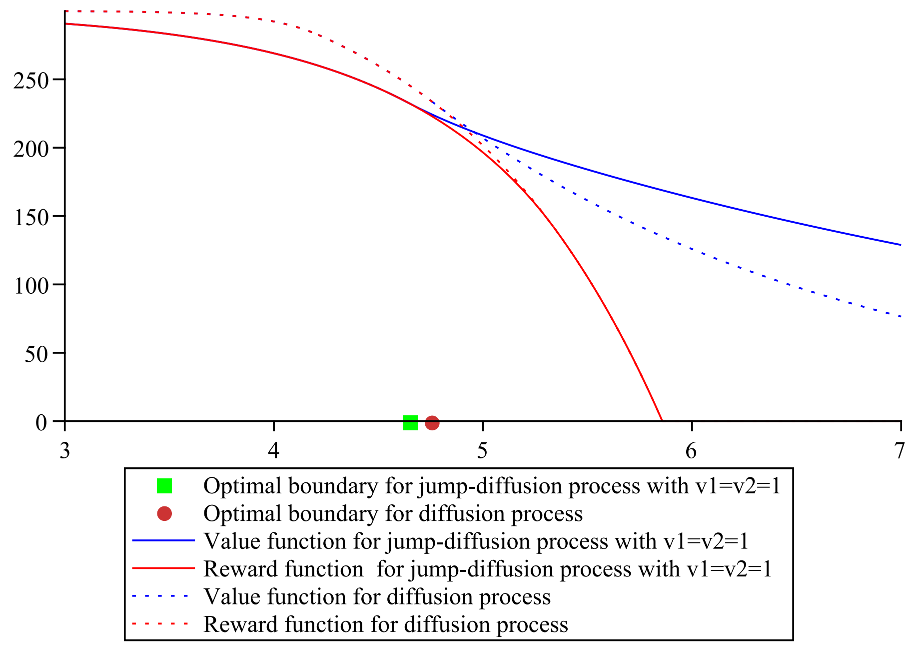

Figure 5: put-on-call options for jump-diffusion process with v1=v2=1 and diffusion process.

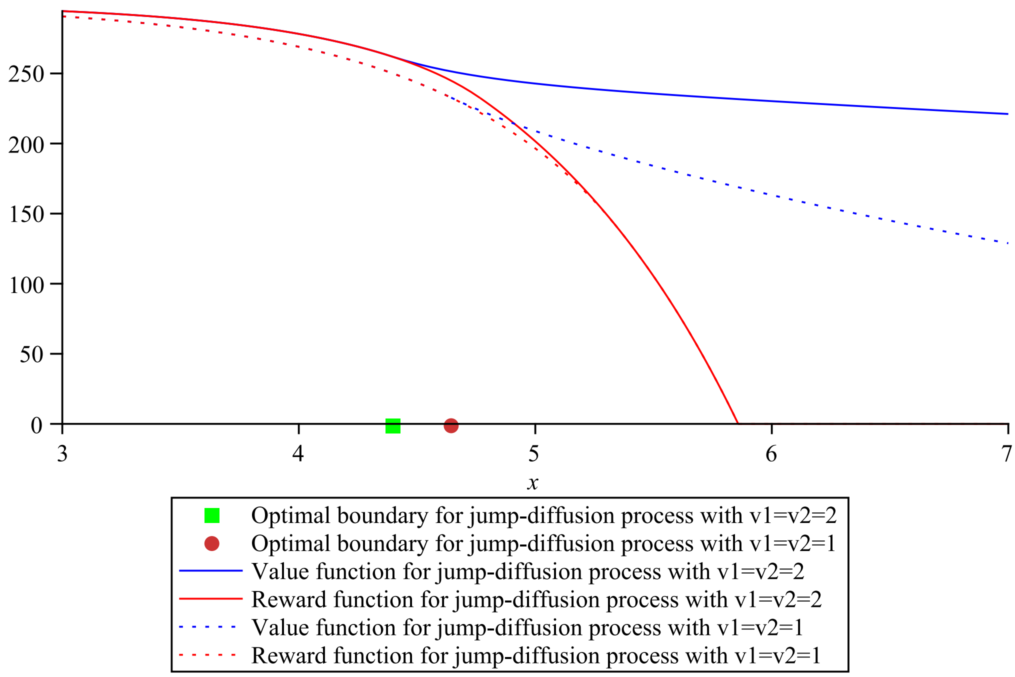

Figure 6: put-on-call options for jump-diffusion process with v1=v2=1 and v1=v2=2.

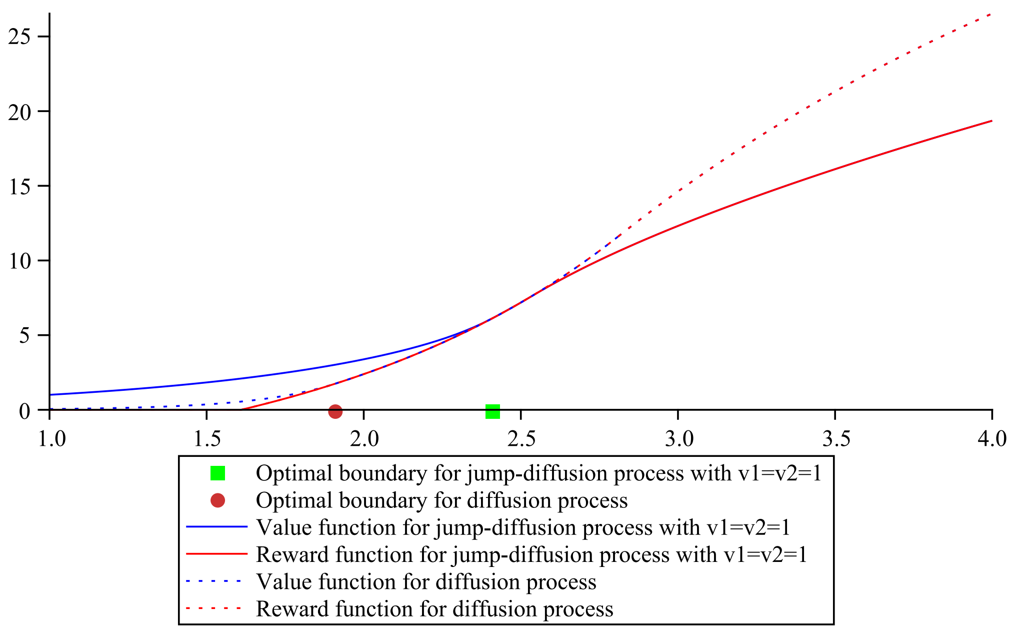

Figure 7: put-on-put options for jump-diffusion process with v1=v2=1 and diffusion process.

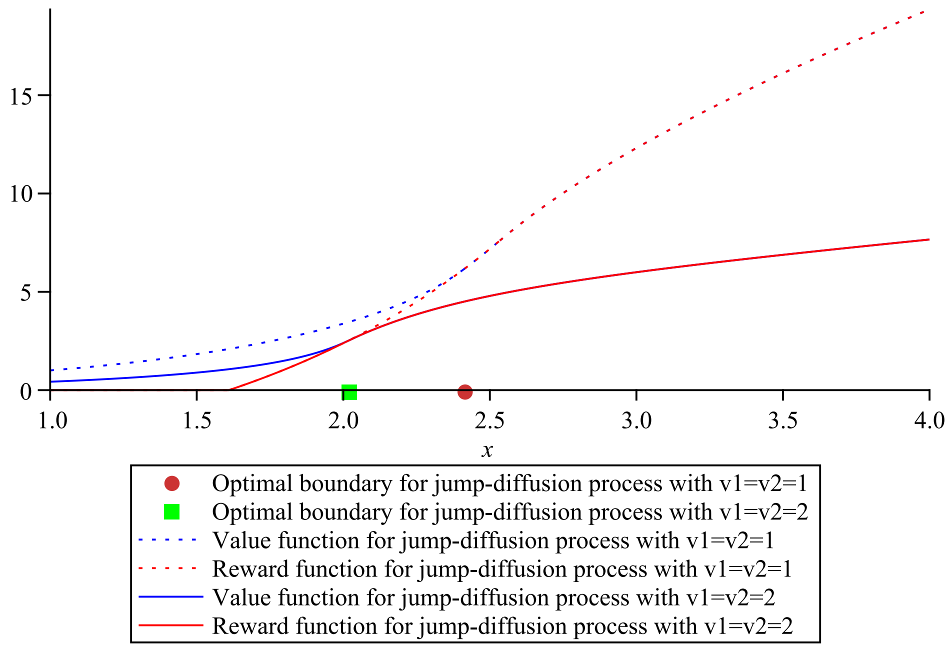

Figure 8: put-on-put options for jump-diffusion process with v1=v2=1 and v1=v2=2.

Table 1: Parameters for Jump-Diffusion Processes

	Diffusion Process	Exponent Jump Diffusion Process	Mixture Exponent Jump Diffusion Process
a	-0.105	-0.105	-0.105
b	0.25	0.25	0.25
λ	-	0.3	0.3
μ	-	0.3	0.3
β_1	-	2.5	2
α_1	-	1.428571429	0.1333333333
β_2	-	-	4
α_2	-	-	0.4166666667
c_{11}	-	1	0.5
c_{21}	-	-	0.5
$\widetilde{c}_{11}$	-	1	0.5
$\widetilde{c}_{21}$	-	-	0.5

Table 2: Compound options

	Diffusion Process	Exponent Jump Diffusion Process	Mixture Exponent Jump Diffusion Process
call-on-call			
K_1	10	10	10
K_2	50	50	50
x_c^*	4.212076578	5.336364625	4.766693426
x_1^*	4.394398135	5.518686182	4.949014981
call-on-put			
K_1	20	20	20
L_2	50	50	50
x_p^*	2.810341610	2.534084694	2.004288173
x_2^*	2.299515986	2.023259069	1.493462547
put-on-call			
L_1	300	300	300
K_2	50	50	50
x_c^*	4.212076578	5.336364625	4.766693426
x_3^*	4.756251761	4.644257544	4.395873979
put-on-put			
L_1	45	45	45
L_2	50	50	50
x_p^*	2.810341610	2.534084694	2.004288173
x_4^*	1.909491485	2.409138962	2.017624207

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國科會補助計畫衍生研發成果推廣資料表

日期:2012/08/29

國科會補助計畫	計畫名稱: 最佳停止問題與套利
	計畫主持人: 許元春
	計畫編號: 100-2115-M-009-006-　　　　學門領域: 機率論

無研發成果推廣資料

100 年度專題研究計畫研究成果彙整表

計畫主持人：許元春			計畫編號：100-2115-M-009-006-				
計畫名稱：最佳停止問題與套利							
成果項目			量化			單位	備註（質化說明：如數個計畫共同成果、成果列為該期刊之封面故事...等）
			實際已達成數（被接受或已發表）	預期總達成數(含實際已達成數)	本計畫實際貢獻百分比		
國內	論文著作	期刊論文	0	0	100%	篇	
		研究報告/技術報告	2	2	100%		
		研討會論文	0	0	100%		
		專書	0	0	100%		
	專利	申請中件數	0	0	100%	件	
		已獲得件數	0	0	100%		
	技術移轉	件數	0	0	100%	件	
		權利金	0	0	100%	千元	
	參與計畫人力（本國籍）	碩士生	2	2	100%	人次	
		博士生	2	2	100%		
		博士後研究員	0	0	100%		
		專任助理	0	0	100%		
國外	論文著作	期刊論文	1	1	100%	篇	
		研究報告/技術報告	0	0	100%		
		研討會論文	0	0	100%		
		專書	0	0	100%	章/本	
	專利	申請中件數	0	0	100%	件	
		已獲得件數	0	0	100%		
	技術移轉	件數	0	0	100%	件	
		權利金	0	0	100%	千元	
	參與計畫人力（外國籍）	碩士生	0	0	100%	人次	
		博士生	0	0	100%		
		博士後研究員	0	0	100%		
		專任助理	0	0	100%		

其他成果 (無法以量化表達之成果如辦理學術活動、獲得獎項、重要國際合作、研究成果國際影響力及其他協助產業技術發展之具體效益事項等，請以文字敘述填列。)	無

	成果項目	量化	名稱或內容性質簡述
科教處計畫加填項目	測驗工具(含質性與量性)	0	
	課程/模組	0	
	電腦及網路系統或工具	0	
	教材	0	
	舉辦之活動/競賽	0	
	研討會/工作坊	0	
	電子報、網站	0	
	計畫成果推廣之參與（閱聽）人數	0	

國科會補助專題研究計畫成果報告自評表

請就研究內容與原計畫相符程度、達成預期目標情況、研究成果之學術或應用價值（簡要敘述成果所代表之意義、價值、影響或進一步發展之可能性）、是否適合在學術期刊發表或申請專利、主要發現或其他有關價值等，作一綜合評估。

1. 請就研究內容與原計畫相符程度、達成預期目標情況作一綜合評估

■達成目標

□未達成目標（請說明，以 100 字為限）

□實驗失敗

□因故實驗中斷

□其他原因

說明：

2. 研究成果在學術期刊發表或申請專利等情形：

論文：■已發表 □未發表之文稿 □撰寫中 □無

專利：□已獲得 □申請中 ■無

技轉：□已技轉 □洽談中 ■無

其他：（以 100 字為限）

3. 請依學術成就、技術創新、社會影響等方面，評估研究成果之學術或應用價值（簡要敘述成果所代表之意義、價值、影響或進一步發展之可能性）（以 500 字為限）

複合選擇權是一種賦予持有方買或賣該標的選擇權權利的商品。在本文裡，我們要探討永續美式複合權在跳躍擴散模型的假設下的評價問題。從 Gapeev 和 Rodosthesnous 的研究可得知，在跳躍擴散模型的假設下，這種起初是雙重最佳停止時間的問題能被分解成一連串的單一最佳停止問題。利用處理單一最佳停止問題常用的平均方法，我們推導出永續美式複合選擇權在雙重指數型跳躍擴散模型下的顯解。