行政院國家科學委員會專題研究計畫 成果報告

連續時間 filtration 的正交分解
研究成果報告(精簡版)

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公 開 資 訊 ： 本計畫可公開查詢

中 華 民 國 101 年 10 月 30 日

中 文 摘 要 ： 在此計畫中我們想考慮一組布朗運動的線性轉換的性質. 與一般考慮情形不同之處在於此組線性轉換為連續型的狀況.當布朗運動經此線性轉換後仍為布朗運動且所生成的filtration 有變小的情形時, 能否將原本的布朗運動的Brownian filtration化成orthogonal decomposition是我們比較感興趣的問題. 此外, 這個orthogonal decomposition是finite亦或是infinite, 遍歷性質如何, 是我們想探討的主題之一.

中文關鍵詞： 布朗運動； 布朗filtration； 遍歷性質； 正交分解.

英 文 摘 要 ： In this project we are mainly concerned with a group of linear transforms of Brownian motion and the related properties. If the resulting stochastic processes are again Brownian motions and whose Brownian filtration is strictly smaller than the original Brownian filtration, we aim to investigate the orthogonal decomposition of the original Brownian filtration. Moreover, the ergodic property of the resulting stochastic processes is also one of our main discussion points.

英文關鍵詞： Brownian motion； Brownian filtration； ergodic property； orthogonal decomposition.

行政院國家科學委員會補助專題研究計畫 ■ 成果報告
□期中進度報告

連續時間 filtration 的正交分解

計畫類別：■個別型計畫 □整合型計畫
計畫編號：NSC 100 － 2115 － M－009－002
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中　華　民　國　年　月　日

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請就研究內容與原計畫相符程度、達成預期目標情況、研究成果之學術或應用價值（簡要敘述成果所代表之意義、價值、影響或進一步發展之可能性）、是否適合在學術期刊發表或申請專利、主要發現或其他有關價值等，作一綜合評估。

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   說明：因 Professor Yor 身體狀況不佳, Yor 與 Strook 之文章一直未完成, 雖然已得到一些小結果, 但尚未到達能發表及形成成果報告的程度. 因此在這段時間我便開始從事另一方面主題的研究. 因此僅能附上這段時間的另一些研究成果.

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In credit risk modeling, Gaussian and Student's t variates arise primarily from the copula method to retain certain correlation structures among defaultable assets. We propose efficient importance sampling algorithms to estimate lower tail probabilities of these two variates in any finite dimension. Variances of importance sampling estimators are shown asymptotically optimal by means of the large deviation theory and a truncation argument. Numerical comparisons with commercial codes, such as mvncdf.m and mvtcdf.m in Matlab, demonstrate robustness and efficiency of our proposed algorithms. Moreover, the flexibility of these algorithms can be seen from an application of probability estimation for the nth-to-default, i.e., the nth order statistic, given a credit portfolio.

Efficient importance sampling for estimating lower tail probabilities under Gaussian and Student's t distributions*

Chuan-Hsiang Han †

Ching-Tang Wu ‡

October 30, 2012

Abstract

In credit risk modeling, Gaussian and Student's t variates arise primarily from the copula method to retain certain correlation structures among defaultable assets. We propose efficient importance sampling algorithms to estimate lower tail probabilities of these two variates in any finite dimension. Variances of importance sampling estimators are shown asymptotically optimal by means of the large deviation theory and a truncation argument. Numerical comparisons with commercial codes, such as mvncdf.m and mvtcdf.m in Matlab, demonstrate robustness and efficiency of our proposed algorithms. Moreover, the flexibility of these algorithms can be seen from an application of probability estimation for the nth-to-default, i.e., the nth order statistic, given a credit portfolio.

Keywords: copula method, importance sampling, variance analysis, lower tail probability, nth-to-default probability.

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1 Introduction

Recent financial events and studies in credit risk have addressed the importance of high-dimensional computation. A credit portfolio may consist of a large amount of defaultable assets including loans, mortgages, credit derivatives such as credit default swaps and collateral debt obligations, etc. The probability that multiple assets are defaulted at the same time can be used as a measure of extreme risks or lower tail dependence [5]. This joint default probability is also fundamental to predict possible losses [12] and evaluate multi-name credit derivatives [11].

The study of default time correlations is a key subject in credit risk modeling. A detailed account for static and dynamic correlation structures and their estimation methods can be found in [5, 15] and references therein. Recently, Malliavin and Mancino [16] analyze a nonparametric Fourier transform method for estimation of volatilities and correlations under semimartingale models in continuous time. Among all these correlation structures for default times, the copula method [3] has been widely used since last decade. Its popularity is due to the statistical flexibility to handle the comovement between markets, risk factors, etc [3], as opposed to, for instance, another way of using the first passage time under the structural-form models [2, 13]. In this current study, we consider estimation for the probability of joint default under Gaussian and Student's t copula models. This estimation problem is then properly formulated as a lower tail probability under multivariate normal and multivariate Student's t distributions on the assumption that the cumulative distribution function (cdf) of each default time is invertible. Monte Carlo simulation is suitable for high-dimensional computation because its convergence rate is independent of dimension. To improve its accuracy, variance reduction [8, 14] is often considered.

Importance sampling [1] is known as one crucial technique of variance reduction to estimate rare event probabilities. Based on the exponential twist for a selection of measure change, Glasserman et al. [6, 7] studied importance sampling algorithms for portfolios risk management. Market asset returns are assumed to be distributed by either multivariate normal or multivariate Student's t. As a result of portfolio losses using the delta-gamma approximation, their exponential twisting parameters are one-dimensional. In contrast to real-valued portfolio losses studied in [6, 7], our estimation for lower tail probability deals with high-dimensional problems. We apply the exponential twisting to derive importance sampling algorithms, but the total number of twisting parameters is the same as the total number of defaultable assets.

The vector of exponential twisting parameters used in our importance sampling algorithms satisfies a possibly large linear system, which is associated with the covariance matrix defined in the multivariate normal or Student's t distribution. To explore the performance of these stochastic algorithms, we conduct a variance analysis by means of the large deviation theory and a truncation argument. Decay rates of the first and second moments of estimators associated with importance sampling are approximated when tail boundaries are scaled into rare event regimes. We show in an asymptotic sense that the proposed importance sampling algorithms are indeed optimal. In simulation terms, these algorithms are efficient [1] because their variance are zero asymptotically.

From the numerical integration point of view, it is a classical problem to calculate the cdf value as a multiple integral when the joint density function is explicit. In cases of multivariate normal and Student's t, Genz and Brentz [9] developed a Quasi Monte Carlo method to calculate cdf values. Their method were then commercialized as computational schemes. For example in Matlab, mvncdf.m and mvtcdf.m are able to compute cdf values for multivariate normal and Student's t, respectively. Hence, our importance sampling algorithms for estimation of lower tail probabilities can be viewed as stochastic alternatives to these classical schemes. Numerical comparisons are performed in this paper.

To demonstrate the flexibility of our algorithms, we show an application for estimation of the loss density function given a credit portfolio. This can be viewed as a static version of the counter-part dynamical model investigated by Carmona et al. [2]. They studied the loss function estimation problem by means of the interacting particles system under a first passage time model. See also [10] for estimating joint default probabilities under the classical first passage time problem in finance by means of an importance sampling technique. Note that in the later study, the lower tail probability cannot be formulated as an explicit integral problem because the joint density of the hitting times of correlated Brownian motions are not yet known explicitly in high dimension.

The organization of this paper is as follows. Section 2 provides motivation for estimation of lower tail probabilities under multivariate normal and Student's t from the copula method in credit risk. Section 3 presents derivations of our proposed importance sampling estimators by exponential twist. Section 4 and 5 present several estimation results of decay rates for the second moments of proposed importance sampling estimators, and demonstrate numerical performance and their comparisons with Matlab codes mvncdf.m and mvtcdf.m. Section 6 presents some extends of our importance sampling algorithms to estimate loss density functions of some credit portfolios.

2 Lower Tail Probability Estimation

Let d be the total number of defautable assets. We denote by τ_i the default time of the ith asset, $1 \leq i \leq d$, and $F_i(t), 0 \leq t < \infty$, its cdf. Assuming that each $F_i(t)$ is strictly monotone, the ith default time can be sampled from

$$\tau_i = F_i^{-1}(U_i), \tag{1}$$

where each U_i is a uniform-$[0,1]$ random variable. The copula method [3] is useful to provide various correlation structures for these uniform variates $(U_1, \cdots, U_d)$. Next, we consider Gaussian copula and Student's t copula, and formulate the probability of multiple defaults to the probability of lower tail.

A Gaussian copula model specifies each uniform variate U_i by

$$U_i = \Phi_i(Z_i) \tag{2}$$

for $1 \leq i \leq d$. The vector $(Z_1, \cdots, Z_d)^T$ is distributed by a multivariate normal and T denotes the transpose. Each Φ_i denotes the distribution function of the ith normal variate Z_i. Hence, the correlation between default times τ_i and $\tau_j, i \neq j$, depends upon the correlation between normal variates Z_i and Z_j via composite functions $F_i^{-1}\Phi_i$ and $F_j^{-1}\Phi_j$. Suppose that each $Z_i, 1 \leq i \leq d$, admits a special decomposition:

$$Z_i = \rho_i W_0 + \sqrt{1 - \rho_i^2}\, W_i, \tag{3}$$

where W_0 is the common factor and each W_i is a marginal factor. These $W_0, W_1, \cdots, W_d$ are assumed to be one-dimensional *i.i.d.* standard normal random variables with each $|\rho_i| \leq 1$. A Gaussian factor copula model is defined to satisfy the decomposition (3). It is particularly useful for model reduction because the total number of parameters in the covariance matrix are reduced from the order of d^2 to d, as compared to ordinary Gaussian copula models. This is beneficial to statistical estimation. See [3] for detailed discussions.

Analogously, a Student's t copula model specifies $U_i = t_\nu(S_i)$, where each $S_i, i = 1, \cdots, d$, denotes a univariate Student's t with the degree of freedom ν, and t_ν denotes its cdf. The multivariate Student's t $S = (S_1, \cdots, S_d)^T$ considered here satisfies the decomposition $S = Z/\sqrt{\frac{Y}{\nu}}$, where Z denotes a centered multivariate normal with the covariance matrix

Σ in d dimension and Y denotes a univariate Chi square with the degree of freedom ν. The cdf of S is denoted by $t_{\Sigma,\nu}$. When each Z_i has a similar decomposition as (3), this model is called Student's t factor copula model. Some other definitions of multivariate Student's t can be found in [18].

Given a set of fixed time horizon $(T_1, \cdots, T_d)$ for default times $(\tau_1, \cdots, \tau_d)$ respectively, the joint default event is defined as $\Pi_{i=1}^d \mathbb{I}(\tau_i \leq T_i)$. Substituting (2) into (1), each default event $\{\tau_i < T_i\}$ can be characterized by $\{Z_i < c_i\}$ given $c_i := \Phi_i^{-1}(F_i(T_i))$. Under a Gaussian copula model, the probability of joint default is equal to $P = \mathbb{E}\left\{\Pi_{i=1}^d \mathbb{I}(Z_i < c_i)\right\}$, or in vector form

$$P = \mathbb{E}\left\{\mathbb{I}(Z < C)\right\}, \tag{4}$$

where $Z = (Z_1, Z_2, \cdots, Z_d)^T$ and $C = (c_1, c_2, \cdots, c_d)^T$. Since the density function of a multivariate normal is known explicitly, this lower tail probability can be presented as an integral form:

$$P = \int_{-\infty}^{c_1} \cdots \int_{-\infty}^{c_d} \frac{1}{(2\pi)^{\frac{d}{2}} |\Sigma|^{\frac{1}{2}}} \exp\left(-\frac{1}{2} z^T \Sigma^{-1} z\right) dz_d \cdots dz_1, \tag{5}$$

where we denote by $z = (z_1, \cdots, z_d)^T$.

Likewise, in the case of Student's t copula model, the probability of joint default can be defined as

$$P = \mathbb{E}\left\{\mathbb{I}(S < C)\right\}, \tag{6}$$

where $S = (S_1, S_2, \cdots, S_d)^T$ is a multivariate Student's t with covariance matrix Σ and with the degree of freedom $\nu > 0$, and $C = (c_1, \cdots, c_d)^T$ denotes the (default) threshold vector with each entry $c_i = t_\nu^{-1}(F_i(T_i))$. Given the joint density function of S, the integral form of this lower tail probability is known as

$$P = \int_{-\infty}^{c_1} \cdots \int_{-\infty}^{c_d} \frac{\Gamma(\frac{\nu+d}{2}) |\Sigma|^{\frac{1}{2}}}{\Gamma(\frac{\nu}{2}) (\nu\pi)^{\frac{d}{2}}} \left(1 + \frac{1}{\nu} z^T \Sigma^{-1} z\right)^{-\frac{\nu+d}{2}} dz_d \cdots dz_1, \tag{7}$$

where we denote by $z = (z_1, \cdots, z_d)^T$.

3 Importance Sampling by Exponential Twist

Here we present a method of importance sampling, namely exponential twist, for estimation of probabilities defined in (4,6), or equivalently computation of cumulative integral functions (5,7), respectively, on the left tail regions. The key assumption is that the joint moment generating function of underlying variates or their transformations must be explicitly known. We now briefly review the importance sampling by exponential twist to estimate the probability $\mathbb{P}(Z < C) = \mathbb{E}\left\{\mathbb{I}(Z < C)\right\}$ in the following.

Suppose that under a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, the multivariate $Z \in \Re^d$ has a density function f. For simplicity, we assume that Z is continuous, $f(z) > 0$ for $z \in \Re^d$, and its moment generating function denoted by $M_Z(\mu)$ exists, where $\mu = (\mu_1, \cdots, \mu_d)^T$ denotes a vector of parameters in the generating function. The exponential twist imposes a new density function defined by

$$f_\mu(z) = \frac{\exp(\mu z) f(z)}{M_Z(\mu)} \tag{8}$$

for measure change. Under the new measure $\mathbb{P}_\mu$ defined from the Rodan-Nykodym derivative $d\mathbb{P}/d\mathbb{P}_\mu = \exp(\mu z)/M_Z(\mu)$, the lower tail probability $\mathbb{P}(Z < C)$ can be expressed by

$$P_1 = \mathbb{E}_\mu \left\{ \mathbb{I}(Z < C) \frac{f(Z)}{f_\mu(Z)} \right\}. \tag{9}$$

Let $P_2(\mu)$ denote the second moment of $\mathbb{I}(Z \leq C) f(Z)/f_\mu(Z)$ under the new measure $\mathbb{P}_\mu$. That is $P_2(\mu) = \mathbb{E}_\mu \left\{ \mathbb{I}(Z < C) f^2(Z)/f_\mu^2(Z) \right\}$. It is easy to see that $P_2(\mu) = \mathbb{E}\left\{ \mathbb{I}(Z < C) f(Z)/f_\mu(Z) \right\}$ under the original probability measure $\mathbb{P}$. Substituting the choice of $f_\mu(z)$ into $P_2(\mu)$, we obtain

$$\begin{aligned} P_2(\mu) &= M_Z(\mu) \mathbb{E}\left\{ \mathbb{I}(Z < C) \exp(-\mu^T Z) \right\} \\ &\leq M_Z(\mu) \mathbb{E}\left\{ \mathbb{I}(Z < C) \exp(-\mu^T C) \right\} \\ &\leq M_Z(\mu) \exp(-\mu^T C), \end{aligned} \tag{10}$$

in which we assume that all c_i and μ_i for $i = 1, \cdots, d$ are negative numbers for the first inequality to be held. Since the indicator function is bounded above by 1, the second inequality is satisfied.

We intend to minimize the variance of $\mathbb{I}(Z < C) f(Z)/f_\mu(Z)$ shown in (18) over μ. This task is reduced to minimize the second moment $P_2(\mu)$ because P_1 is actually μ-independent. It is a challenging problem to solve for the minimizer of $P_2(\mu)$ particularly in high-dimensional cases. When the moment generating function is in exponential form, it may appear that minimizing the logarithm of the upper bound (10) becomes tractable. Thus, it provides a candidate for importance sampling. According to the first order condition, each partial derivative must be zero to solve for μ. Let $\mu^* = (\mu_1^*, \cdots, \mu_d^*)^T$ admits the solution of $\nabla \ln(M_Z(\mu^*)) - \mu^{*T} C = 0$ or equivalently for each component $i \in \{1, \cdots, d\}$

$$\frac{1}{M_Z(\mu^*)} \frac{\partial M_Z(\mu^*)}{\partial \mu_i^*} = c_i. \tag{11}$$

In both multivariate normal and transformed Student's t, it turns out these equations can be solved explicitly for μ^*.

It follows that the expected value of Z under the new probability measure $\mathbb{P}_{\mu^*}$ is exactly the threshold vector C. This is confirmed by a direct calculation $I\!E_{\mu^*}(Z_i) = \int_{-\infty}^{\infty} z_i \, f_{\mu^*}(z_i) \, dz_i$ using $f_{\mu^*}(z)$ defined in (8) such that

$$\begin{aligned} I\!E_{\mu^*}(Z_i) &= \frac{1}{M_Z(\mu^*)} \frac{\partial M_Z(\mu^*)}{\partial \mu_i^*} \\ &= c_i \end{aligned} \tag{12}$$

is obtained. The last equality comes from (11). Equation (12) reveals that "the expected value of X under the new probability measure $\mathbb{P}_{\mu^*}$ is equal to the threshold C." From the simulation point of view, this result is appealing because the rare event $\{X < C\}$ under the original probability measure is no longer rare under this new measure $\mathbb{P}_{\mu^*}$ associated with the density function f_{μ^*}. However, it still requires a qualitative check on whether this choice of measure change is optimal or not.

3.1 Importance Sampling for Multivariate Normal Distribution

Since the moment generating function of $Z \sim \mathcal{N}(0, \Sigma)$ is $M_Z(\mu) = \exp\left(\frac{1}{2}\mu^T \Sigma \mu\right)$, it is easy to solve for an optimal candidate

$$\Sigma\, \mu^* = C, \tag{13}$$

which results from Equation (11) for each $i = 1, \cdots, d$. In order to faciliate numerical comparisons, a pseudo algorithm for estimation of a lower tail probability under a centered multivariate normal Z is provided below.

Algorithm 1: Estimation of lower tail probability under centered multivariate Normal

1. Given the distribution $Z \sim \mathcal{N}(0, \Sigma)$ and the lower threshold $C < 0$, compute $\mu^* = \Sigma^{-1} C$.
2. For each independent ith replication, $i = 1, \cdots, m$,
   (a) Generate $Z^{(i)} = (Z_1^{(i)}, \cdots, Z_d^{(i)})^T$ from $\mathcal{N}(C, \Sigma)$.
   (b) Evaluate $M_Z(\mu^*) \exp\left(-\mu^{*T} Z^{(i)}\right) \mathbb{I}\{Z^{(i)} < C\}$.
3. Compute the average of samples generated from (b) in Step 2.

3.1.1 Numerical Comparison with mvncdf.m

Homogeneous experiments are conducted to estimate the lower tail probability (4) given a factor decomposition (3). That is, model parameters ρs in the convariance matrix Σ and cs in the threshold vector C are chosen the same. For example in our numerical experiments, we set $\rho = \rho_i = 0.5$ and $c = c_i = -2$ for $i = 1, \cdots, d$. Numerical performance between the basic Monte Carlo method, our importance sampling scheme, and a Matlab scheme **mvncdf.m** are compared. This Matlab code is based on a quasi Monte Carlo method developed in [9]. In Table 1, the dimension number d varies from 5 to 25, then 30, 50, up to 125. (The maximal number 25 is set in the Matlab code, but there is no dimension restriction for our importance sampling scheme.) Numerics, including sample means and standard errors, generated from the proposed importance sampling scheme are roughly at the same order of accuracy compared with those calculated from Matlab. Matlab often performs better before d=? and the importance sampling outperforms for rest cases. The total number of simulation for each numerical experiment is 75000. Note that on average the computing time of our algorithm is ?% less than the Matlab code. This is probably due to repeated usage of a sequences of inverse functions of normal integrals in the Matlab algorithm, while we only solves for a single linear system (13).

3.2 Importance Sampling for Multivariate Student's t Distribution

As discussed in (6), recall the setup of the estimation problem $P = \mathbb{P}(S < C)$, where $S = (S_1, \cdots, S_d)^T \sim t_{\Sigma,\nu}$ and $C = (c_1, \cdots, c_d)^T$ with each $c_i = t_\nu^{-1}(F_i(T_i))$ for $i = 1, \cdots, d$. That is, the factorization $S = Z/\sqrt{\frac{Y}{\nu}}$ is allowed, where Z denotes a centered multivariate normal with the covariance matrix Σ in d dimension, and Y denotes a univariate Chi square with the degree of freedom ν. Let $AA^T = \Sigma$ by the Cholesky decomposition such that $S = AW/\sqrt{\frac{Y}{\nu}}$, where $W = (W_1, \cdots, W_d)^T$ is $\mathcal{N}(0, I_d)$-distributed. We define a transformation $\tilde{X} = Y/\nu\,(S-C)$. The tail event $\{S < C\}$ is equal to $\left\{\tilde{X} < 0\right\}$ because the Chi square Y is positive almost surely. Hence, the tail probability becomes $P = \mathbb{P}(\tilde{X} < 0)$. As discussed in the derivation of importance sampling by exponential twist, we manage to minimize the upper bound $M_{\tilde{X}}(\mu)$, similar to (10) with $C = 0$, of the second moment $P_2(\mu)$. We shall first investigate the moment generating function of $\tilde{X}$.

Lemma 1. *Given a symmetric and positive definite covariance matrix Σ and the degree of freedom ν, assume that the parameter vector $\mu \in \Re^d$ satisfy $\frac{\mu^T\Sigma\mu - 2\mu^T C}{\nu} < 1$. The moment generating function of $\tilde{X}$ is*

$$M_{\tilde{X}}(\mu) = \left(1 - \frac{1}{\nu}\mu^T\Sigma\mu + \frac{2}{\nu}\mu^T C\right)^{-\nu/2}.$$

Proof: Recall the transformation $\tilde{X} = \frac{Y}{\nu}(S - C) = -\frac{Y}{\nu}C + A\left(\sqrt{\frac{Y}{\nu}}W\right)$. Conditional on the Chi square random variable Y, the moment generating function of $\tilde{X}$ is

$$\begin{aligned}
M_{\tilde{X}}(\mu) &= \mathbb{E}\left[\mathbb{E}\left[\exp\left(\mu^T\tilde{X}\right)|Y\right]\right] \\
&= \mathbb{E}\left[\exp\left(-\frac{Y}{\nu}\mu^T C\right) + \mathbb{E}\left[\exp\left(\sqrt{\frac{Y}{\nu}}\mu^T AW\right)|Y\right]\right] \\
&= \mathbb{E}\left[\exp\left(-Y\frac{\mu^T C}{\nu} + Y\frac{\mu^T\Sigma\mu}{2\nu}\right)\right] \\
&= \left(1 - \frac{1}{\nu}\mu^T\Sigma\mu + \frac{2}{\nu}\mu^T C\right)^{-\nu/2}.
\end{aligned}$$

We have used the moment generating functions of the multinormal variate W and the Chi square Y separately in the last two lines. □

From this explicit result, we are able to minimize the logarithm of $M_{\tilde{X}}(\mu)$. By the first order condition $\nabla \ln M_{\tilde{X}}(\mu) = 0$, it is a direct calculation to obtain the solution $\mu^* = \Sigma^{-1}C$.

Table 1: Estimation of lower tail probability under Gaussian copula factor model.

	Basic MC		Importance Sampling		Quasi MC mvncdf.m	
d	$Mean$	SE	$Mean$	SE	$Mean$	SE
5	2.67E-05	1.89E-05	1.37E-05	3.31E-07	1.40E-05	4.31E-08
6	-	-	4.68E-06	1.33E-07	4.71E-06	3.86E-08
7	-	-	1.89E-06	6.58E-08	1.86E-06	1.60E-08
8	-	-	7.80E-07	3.02E-08	8.11E-07	1.13E-08
9	-	-	4.00E-07	1.92E-08	3.70E-07	1.28E-08
10	-	-	2.12E-07	1.08E-08	1.98E-07	4.41E-09
11	-	-	1.06E-07	5.75E-09	1.16E-07	3.92E-09
12	-	-	6.70E-08	4.42E-09	6.34E-08	1.74E-09
13	-	-	4.02E-08	2.97E-09	3.78E-08	8.78E-10
14	-	-	2.38E-08	1.75E-09	2.25E-08	8.65E-10
15	-	-	1.59E-08	1.30E-09	1.49E-08	5.20E-10
16	-	-	1.12E-08	1.25E-09	9.42E-09	2.63E-10
17	-	-	7.21E-09	6.48E-10	6.84E-09	6.99E-10
18	-	-	4.05E-09	3.80E-10	4.57E-09	2.88E-10
19	-	-	3.65E-09	3.45E-10	3.42E-09	3.98E-10
20	-	-	2.41E-09	2.64E-10	2.11E-09	1.18E-10
21	-	-	2.08E-09	2.21E-10	1.72E-09	1.45E-10
22	-	-	1.62E-09	2.00E-10	1.06E-09	8.50E-11
23	-	-	1.06E-09	1.71E-10	1.00E-09	8.33E-11
24	-	-	7.18E-10	7.79E-11	8.13E-10	1.41E-10
25	-	-	5.64E-10	9.36E-11	5.39E-10	4.35E-11
30	-	-	2.01E-10	4.03E-11	-	-
50	-	-	3.84E-12	1.53E-12	-	-
75	-	-	4.18E-13	1.83E-13	-	-
100	-	-	6.99E-14	3.41E-14	-	-

Averaged CPU time in seconds: 1.47E-01, 1.82E-01, 1.69E-01, respectively, without dimensions of 30 and beyond.

This result suggests a possibility to change measure. Note that the optimal μ^* fulfills automatically the assumptions $\left(\mu^{*T}\Sigma\mu - 2\mu^{*T}C\right)/\nu = -\left(C^T\Sigma^{-1}C\right)/\nu < 1$ in Lemma 1 because the covariance matrix Σ is positive definite. Thus the existence of the moment generating function $M_{\tilde{X}}(\mu^*)$ is assured.

For the purpose of executing simulations, we need to understand what distributions of underlying random variables including Y and W are under a new measure $\mathbb{P}_\mu$.

Lemma 2. *Under the prescribed measure, the original χ^2_ν-distributed Y becomes a Gamma random variable with with shape parameter $\nu/2$ and scale parameter $2/(1-2\alpha(\mu))$, where $\alpha(\mu) = (\mu^T\Sigma\mu - 2\mu^{\mathbf{T}}\mathbf{C})/(2\nu)$, and the original standard multivariate normal $\mathbf{W}$ becomes a conditional normal with mean $-\sqrt{\frac{Y}{\nu}}A^T\mu$ and variance I_d given Y.*

$$\begin{aligned} Y &\sim \Gamma\left(\frac{\nu}{2}, \frac{2}{1-2\alpha(\mu)}\right), \\ W|Y &\sim \mathcal{N}\left(-\sqrt{\frac{Y}{\nu}}A^T\mu, I_d\right). \end{aligned}$$

Proof: From the definition of $f_Y^\mu(y) = \frac{d}{dy}\mathbb{E}_\mu\left[\mathbb{I}\left(Y \le y\right)\right]$ and the choice of the Radon-Nikodym derivative $d\mathbb{P}_\mu/d\mathbb{P} = \exp(\mu^T\tilde{\mathbf{X}})/M_{\tilde{\mathbf{X}}}(\mu)$

$$f_Y^\mu(y) = \frac{d}{dy}\mathbb{E}\left[\frac{\mathbb{I}\left(Y \le y\right)}{M_{\tilde{\mathbf{X}}}(\mu)}\exp\left(-\frac{Y}{\nu}\mu^T\mathbf{C} + \sqrt{\frac{Y}{\nu}}\mu^T A\mathbf{W}\right)\right].$$

Again, using condition on Y and the moment generating function of $\mathbf{W}$,

$$f_Y^\mu(y) = \frac{d}{dy}\mathbb{E}\left[\frac{\mathbb{I}\left(Y \le y\right)}{M_{\tilde{\mathbf{X}}}(\mu)}\exp\left(-\frac{Y}{\nu}\mu^T\mathbf{C} + \frac{Y}{2\nu}\mu^T\Sigma\mu\right)\right]$$

is obtained. A direct calculation of this derivative gives $f_Y^\mu(y) = \dfrac{\exp\left(\alpha(\mu)y\right)}{(1-2\alpha(\mu))^{-\nu/2}}f_Y(y)$, where $\alpha(\mu) = \frac{\boldsymbol{\mu^T\Sigma\mu - 2\mu^T C}}{\mathbf{2\nu}}$. By the explicit density function of $Y \sim \chi^2_\nu$, we derive the density of Y

$$f_Y^\mu(y) = \left(\frac{2}{1-2\alpha(\mu)}\right)^{-\nu/2}\frac{y^{(\nu-2)/2}}{\Gamma(\nu/2)}\exp\left(\frac{-y}{2/(1-2\alpha(\mu))}\right),$$

under the new measure $\mathbb{P}_\mu$.

Given Y, the components of $(W_1, \cdots, W_d)$ are independent normal random variables, and the mean of W changes to $-\sqrt{\frac{Y}{\nu}}A^T\mu$. These can be obtained from the following separation

$$\frac{e^{\mu^T\tilde{X}}}{M_{\tilde{X}}(\mu)} = \frac{e^{\alpha(\mu)Y}}{M_Y\left(\alpha(\mu)\right)} \cdot \frac{f^\mu_{W|Y}}{f_W},$$

so that the conditional density

$$f^{\mu}_{\mathbf{W}|\mathbf{Y}}(\mathbf{w}) = f_W(w)\exp\left(\mu^T\tilde{X} - \alpha(\mu)Y\right)$$
$$= \frac{1}{(2\pi)^{d/2}}\exp\left(-\frac{\left(w-\sqrt{\frac{Y}{\nu}}A^T\mu\right)^T\left(w-\sqrt{\frac{Y}{\nu}}A^T\mu\right)}{2}\right)$$

by using definitions of $\tilde{X}, \alpha(\mu)$, and the completion of squares. □

Algorithm 2: Estimation of lower tail probability under multivariate Student's t

1. Given the distribution $S \sim t_{\Sigma,\nu}$ and the lower threshold $C < 0$, compute $\mu^* = \Sigma^{-1}C$
2. For each independent ith replication, $i = 1, \cdots, m$,
   (a) Generate $Y^{(i)}$ from $\Gamma\left(\frac{\nu}{2}, \frac{2}{1-2\alpha(\mu^*)}\right)$, where $\alpha(\mu^*) = \frac{\mu^{*T}\Sigma\mu^*}{2\nu} - \frac{\mu^{*T}C}{\nu}$.
   (b) Given such $Y^{(i)}$, generate $W^{(i)}$ from $\mathcal{N}(-\sqrt{Y/\nu}A^{-1}C, \mathbb{I}_{d\times d})$
   (c) Set $\tilde{X}^{(i)} = Y^{(i)}/\nu(A\,W^{(i)}/\sqrt{\frac{Y^{(i)}}{\nu}} - C)$
   (d) Evaluate the ith sample value $M_{\tilde{X}}(\mu^*)e^{-\mu^{*T}\tilde{X}^{(i)}}\mathbb{I}\{\tilde{X}^{(i)} < 0\}$
3. Compute the average of samples generated from (d) in Step 2.

3.2.1 Numerical Comparison with mvtcdf.m

We use Algorithm 2 to compare with the Matlab code **mvtcdf.m**. Numerics are reported in Table 2 for a small degree of freedom 3 because this order one regime is considered to generate heavy tails. Homogeneous model parameters are chosen as $\rho = \rho_i = 0.5$ in the convariance matrix Σ and $c = c_i = -2$ for $i = 1, \cdots, d$ in the threshold vector C. The total number of simulation for each numerical experiment is 75000. We denote IS by our proposed importance sampling. It can be seen that even in the regime of a small degree of freedom and low dimension, our importance sampling outperforms, and its averaged computing time is ?% less than Matlab.

4 Asymptotic Variance Analysis for Multivariate Normal

Next we focus on the theory of the estimation problem (4) in high dimension. Let matrix A satisfy the Cholesky decomposition of a covariance matrix $\Sigma = AA^T$ so that $AW \sim \mathcal{N}(0, \Sigma)$, and $C < 0$ be the threshold vector. A scale α is introduced to define the scaled lower tail probability by

$$P_1(\alpha) = \mathbb{E}\left\{\mathbb{I}(AW < \sqrt{\alpha}\,C)\right\}, \tag{14}$$

where W is assumed to be d-dimensional independent normal variate. As discussed in Section 3.1, let $\mathbb{P}_\mu$ be the new probability measure so that X is distributed by $\mathcal{N}(\mu, \Sigma)$

Table 2: Estimation of lower tail probability under Student-T factor copula model.

	Basic MC		Importance Sampling		Quasi MC mvtcdf.m	
d	$Mean$	SE	$Mean$	SE	$Mean$	SE
5	2.25E-03	1.73E-04	1.91E-03	3.00E-05	1.90E-03	2.02E-05
6	1.05E-03	1.18E-04	1.16E-03	2.19E-05	1.18E-03	2.63E-05
7	9.33E-04	1.12E-04	7.83E-04	1.72E-05	7.47E-04	1.41E-05
8	4.53E-04	7.77E-05	4.96E-04	1.30E-05	5.21E-04	2.50E-05
9	4.27E-04	7.54E-05	3.63E-04	1.08E-05	3.55E-04	1.27E-05
10	2.93E-04	6.25E-05	2.84E-04	9.16E-06	2.69E-04	8.42E-06
11	2.27E-04	5.50E-05	2.02E-04	7.45E-06	1.88E-04	8.01E-06
12	1.87E-04	4.99E-05	1.57E-04	6.40E-06	1.56E-04	7.72E-06
13	1.47E-04	4.42E-05	1.28E-04	5.96E-06	1.21E-04	6.71E-06
14	8.00E-05	3.27E-05	1.21E-04	5.75E-06	1.08E-04	6.67E-06
15	5.33E-05	2.67E-05	8.97E-05	4.75E-06	8.12E-05	4.38E-06
16	1.47E-04	4.42E-05	6.21E-05	3.67E-06	6.81E-05	6.92E-06
17	6.67E-05	2.98E-05	5.39E-05	3.49E-06	6.03E-05	9.26E-06
18	2.67E-05	1.89E-05	4.48E-05	3.15E-06	5.65E-05	7.23E-06
19	1.33E-05	1.33E-05	4.11E-05	2.93E-06	3.86E-05	5.38E-06
20	2.67E-05	1.89E-05	2.94E-05	2.37E-06	3.56E-05	4.90E-06
21	5.33E-05	2.67E-05	2.76E-05	2.30E-06	3.07E-05	4.53E-06
22	4.00E-05	2.31E-05	2.39E-05	2.18E-06	2.53E-05	4.27E-06
23	-	-	2.19E-05	2.22E-06	2.26E-05	4.32E-06
24	4.00E-05	2.31E-05	1.94E-05	2.07E-06	1.47E-05	2.42E-06
25	4.00E-05	2.31E-05	1.71E-05	1.87E-06	1.28E-05	3.17E-06
30	-	-	1.07E-05	1.48E-06	-	-
50	-	-	1.76E-06	4.80E-07	-	-
75	-	-	6.64E-07	3.09E-07	-	-
100	-	-	2.54E-07	1.23E-07	-	-

Averaged CPU time in seconds: 1.60E-01, 3.87E-01, 2.34E-01, respectively, without dimensions of 30 and 50.

under $\mathbb{P}_\mu$ and $P_1(\alpha) = \mathbb{E}_\mu \left\{ \mathbb{I}(AW < \sqrt{\alpha}\, C) \frac{d\mathbb{P}}{d\mathbb{P}_\mu} \right\}$. This shifted mean μ is a vector of size d. The second moment of the weighted random variable is defined as

$$P_2(\alpha) = \mathbb{E}_\mu \left\{ \left(\mathbb{I}(AW < \sqrt{\alpha}\, C) \frac{d\mathbb{P}}{d\mathbb{P}_\mu} \right)^2 \right\}. \tag{15}$$

It is shown below the optimal choice of μ is $\mu^* = \sqrt{\alpha}\, C$ in the asymptotic sense, so that the second moment is approximately the square of the tail probability.

Theorem 1. *Assume that the scale α is a positive number, each element in the vector $C \in \mathbb{R}^d$ is negative, $W \sim \mathcal{N}(0, I_d)$, and the lower triangular matrix A satisfies the Cholesky decomposition of a given covariace matrix $AA^T = \Sigma$. We obtain the following asymptotic approximations:*

$$\lim_{\alpha \to \infty} \frac{1}{\alpha} \log P_2(\alpha) = 2 \lim_{\alpha \to \infty} \frac{1}{\alpha} \log P_1(\alpha) = -C^T \Sigma^{-1} C.$$

That implies that the importance sampling scheme

$$P_1(\alpha) = \mathbb{E}_{\sqrt{\alpha}\, C} \left\{ \mathbb{I}(X < \sqrt{\alpha}\, C) \exp(2\sqrt{\alpha} C^T \Sigma^{-1} X + \alpha C^T \Sigma^{-1} C) \right\},$$

where $X := AW \sim \mathcal{N}(\sqrt{\alpha}\, C, \Sigma)$ to estimate the lower tail probability is asymptotically optimal.

Proof: Note

$$\begin{aligned} \mathbb{P}\left(X < C\right) &= \mathbb{E}_{(0,\cdots,0)}\left[\mathbb{I}\left(X < C\right)\right] \\ &= \mathbb{E}_{(\mu_1,\cdots,\mu_d)}\left[\mathbb{I}\left(AW < C\right) \prod_{i=1}^d e^{-\mu_i W_i + \frac{\mu_i^2}{2}}\right], \end{aligned} \tag{16}$$

where the lowerscript denotes the corresponding probability measure, under which the mean of the underlying random vector is shown.

The second moment of the last equation is

$$\begin{aligned} &\mathbb{E}_{(\mu_1,\cdots,\mu_d)}\left[\mathbb{I}\left(AW < C\right) \prod_{i=1}^d e^{-2\mu_i W_i}\right] \prod_{i=1}^d e^{\mu_i^2} \\ =&\mathbb{E}_{(-\mu_1,\cdots,-\mu_d)}\left[\mathbb{I}\left(AW < C\right)\right] e^{\sum_{i=1}^d \mu_i^2} \\ =&\mathbb{E}_{(0,\cdots,0)}\left[\mathbb{I}\left(A\left(W-\mu\right) < C\right)\right] e^{\sum_{i=1}^d \mu_i^2} \end{aligned}$$

Now we choose $\mu = (\mu_1, \cdots, \mu_d)^T$ which solves $A\mu = C$ so that $\sum_{i=1}^d \mu_i^2 = \mu^T \mu = C^T (A^{-1})^T A^{-1} C = C^T \Sigma^{-1} C$, and the second moment becomes

$$\mathbb{E}_{(0,\cdots,0)}\left[\mathbb{I}\left(AW < 2C\right)\right] e^{C^T \Sigma^{-1} C}. \tag{17}$$

Introducing a rescaled C by $\sqrt{n}C$ for n being a large positive integer. Given this new scaled $\sqrt{n}C$. Let's denote the first and second moments defined in (16) and (17) by $M_1(n)$

and $M_2(n)$, respectively. That is

$$\begin{aligned} M_1(n) &= \mathbb{E}_{(0,\cdots,0)}\left[\mathbb{I}\left(\frac{1}{\sqrt{n}}X < C\right)\right] \\ &= \mathbb{E}_{(0,\cdots,0)}\left[\mathbb{I}\left(\frac{1}{n}\sum_{i=1}^{n} X^{(i)} < C\right)\right], \end{aligned}$$

where $X^{(i)}$ is i.i.d. as X, and

$$M_2(n) = \mathbb{E}_{(0,\cdots,0)}\left[\mathbb{I}\left(\frac{1}{\sqrt{n}}X < 2C\right)\right] e^{nC^T\Sigma^{-1}C}.$$

By the large deviation principle for Gaussian variates [1], we obtain

$$\begin{aligned} \frac{1}{n}\log M_1(n) &\longrightarrow -I(C) = -\frac{1}{2}C^T\Sigma^{-1}C \\ \frac{1}{n}\log M_2(n) &\longrightarrow -I(2C) + C^T\Sigma^{-1}C \\ &= -C^T\Sigma^{-1}C. \end{aligned}$$

Hence $2\lim_{n\to\infty}\frac{1}{n}\log M_1(n) = \lim_{n\to\infty}\frac{1}{n}\log M_2(n)$. That is, we prove that the proposed importance sampling is asymptoticaly optimal or called efficient.

Since $P_1(\alpha)$ is strictly decreasing in α, we can generalize estimates in Theorem 1 to the case with continuous variable α

$$\begin{aligned} \frac{1}{\alpha}\log P_1(\alpha) &\longrightarrow -I(C) = -\frac{1}{2}C^T\Sigma^{-1}C \\ \frac{1}{\alpha}\log P_2(\alpha) &\longrightarrow -I(2C) + C^T\Sigma^{-1}C \\ &= -C^T\Sigma^{-1}C, \end{aligned}$$

when α is large enough.

5 Asymptotic Variance Analysis for Multivariate Student's t

Recall that $S = \frac{AW}{\sqrt{Y/\nu}} = (S_1, S_2, \cdots, S_d)^T$ is a multivariate Student's t random variable with covariance matrix Σ, where $AA^T = \Sigma$ by Cholesky decomposition, $W \sim \mathcal{N}(0, I_d)$ and Y is a Chi-square random variable with degree of freedom ν. Denote $C < 0$ the threshold $n \times 1$-vector. We are interested in estimating the lower tail probability

$$\begin{aligned} P_1 &:= \mathbb{E}\left[\mathbb{I}(S < C)\right] \\ &= \mathbb{E}\left[\mathbb{I}(\frac{AW}{\sqrt{Y/\nu}} < C)\right] \end{aligned} \quad (18)$$

By defining a transformation $\tilde{X} = \frac{Y}{\nu}\left(\frac{AW}{\sqrt{Y/\nu}} - C\right)$, the importance sampling by an exponential change of measure gives the following estimator

$$P_1(C,\nu) = \tilde{\mathbb{E}}\left[I(\tilde{X} < 0)\frac{(1+\frac{C^T\Sigma^{-1}C}{\nu})^{-\nu/2}}{\exp(-\Sigma^{-1}\,C\,\tilde{X})}\right]$$

The second moment of this estimator is

$$P_2^{IS}(C,\nu) = \tilde{\mathbb{E}}\left[I(\tilde{X} < 0)\left(\frac{(1+\frac{C^T\Sigma^{-1}C}{\nu})^{-\nu/2}}{\exp(-\Sigma^{-1}\,C\,\tilde{X})}\right)^2\right],$$

which can be further simplified by

$$P_2^{IS} \;=\; \left(1+\frac{C^T\Sigma^{-1}C}{\nu}\right)^{-\nu/2}\mathbb{E}\left[\exp\left(\frac{3Y}{2\nu}C^T\Sigma^{-1}C\right)I\left(AW < 2\sqrt{\frac{Y}{\nu}}\,C\right)\right]$$

Note that the coefficient in front of the expectation $\left(1+\frac{C^T\Sigma^{-1}C}{\nu}\right)^{-\nu/2}$ is equal to $\mathbb{E}[\exp(-\frac{Y}{2\nu})C^T\Sigma^{-1}C]$, then we can rewrite P_2^{IS} by

$$P_2^{IS} = \mathbb{E}\left[\exp\left(\frac{Y}{\nu}C^T\Sigma^{-1}C\right)I\left(AW < 2\sqrt{\frac{Y}{\nu}}\,C\right)\right] \tag{19}$$

To facilitate the study of small probabilities, we introduce a scale $\alpha > 0$ and multiple it with the default threshold vector $C < 0$ so that when the default threshold vector $\sqrt{\alpha}C$ becomes negatively large, the lower tail probability should be small. One can study its decaying behavior by means of large deviation. Introduce new notations $P_1(\alpha,\nu;C) := P_1(\sqrt{\alpha}\,C,\nu)$ and $P_2^{IS}(\alpha,\nu;C) = P_2^{IS}(\alpha\,C,\nu)$. Then we have the following result.

Theorem 2. *Let $S = \dfrac{AW}{\sqrt{Y/\nu}} = (S_1, S_2, \cdots, S_d)^T$ be a multivariate Student's t random variable with covariance matrix Σ, where $AA^T = \Sigma$ by Cholesky decomposition, $Z \sim N(0, I_d)$ and Y is a Chi-square random variable with degree of freedom ν. Assume that*

(1) the default threshold satisfies $\sqrt{\alpha}\,C < 0$, for α being a positive scale parameter and $C < 0$ the threshold $d \times 1$-vector

(2) the degree of freedom ν satisfy $l\alpha C^T\Sigma^{-1}C/\nu \leq 1$,

then

$$\lim_{\alpha\to\infty}\lim_{\nu\to\infty}\frac{1}{\alpha}\ln P_1(\alpha,\nu;C) = -\frac{C^T\Sigma^{-1}C}{2}.$$
$$\lim_{\alpha\to\infty}\lim_{\nu\to\infty}\frac{1}{\alpha}\ln P_2^{IS}(\alpha,\nu;C) = -C^T\Sigma^{-1}C.$$

Proof: From the result that Student's t converges to one, i.e., $\frac{Y}{\nu} \to 1$, $\lim_{\nu\to\infty} P_1(l,\nu;C) = E[I(AW \leq \sqrt{\alpha}\,C)]$. Further use the result in Theorem 1, we obtain $\lim_{\alpha\to\infty}\lim_{\nu\to\infty}\frac{1}{\alpha}\ln P_1(\alpha,\nu;C) = -\frac{C^T\Sigma^{-1}C}{2}$. Analogously, asymptotic result for P_2^{IS} can be obtained.

This result shows that the proposed importance sampling is indeed asymptotically optimal for variance reduction when the degree of freedom is large enough.

Throughout this section, we say a function $f(\eta) \sim p_\eta$ as η large enough if there exist some positive constants m, M such that for η large enough

$$m \leq \frac{f(\eta)}{p_\eta} \leq M.$$

Next we summarize our main theoretical results.

Theorem 1. *Decay rate estimations:*
1. Decay Rate of the First Moment: fixed $\nu > 0$,

$$\mathbb{E}\left[\mathbb{I}\left(\frac{AW}{\sqrt{Y/\nu}} < \sqrt{\alpha}C\right)\right] \sim \alpha^{-\nu/2},$$

for large α.

2. Decay Rate of Second Moment - Conditional Monte Carlo: fixed $\nu > 0$,

$$\mathbb{E}\left[\left(\mathbb{E}\left[\mathbb{I}\left(AW < \sqrt{\alpha}C\sqrt{\frac{Y}{\nu}}\right)\middle| Y\right]\right)^2\right] \sim \alpha^{-\nu/2},$$

for large α.

3. Decay Rate of Second Moment - Importance Sampling: fixed $\nu > 0$,

$$\mathbb{E}\left[\exp\left(\frac{\alpha Y}{\nu}C^T\Sigma^{-1}C\right)\mathbb{I}\left(AW < 2\frac{\alpha Y}{\nu}C\right)\right] \sim \alpha^{-\nu},$$

for large α.

Theorem 2. *Decay rate estimations: For* $\frac{\alpha}{\nu}$ *large enough, there exist constant* $E_1, E_2 > 1$ *such that*
1. Decay Rate of the First Moment:

$$\frac{1}{\nu}\, E_1^{-\nu/2} \leq \mathbb{E}\left[\mathbb{I}\left(\frac{AW}{\sqrt{Y/\nu}} < \sqrt{\alpha}\, C\right)\right] \leq E_2^{-\nu/2}.$$

2. Decay Rate of Second Moment - Conditional Monte Carlo:

$$\frac{1}{\nu}\, E_1^{-\nu/2} \leq \mathbb{E}\left[\left(\mathbb{E}\left[\mathbb{I}\left(AW < \sqrt{\frac{\alpha Y}{\nu}}\, C\right)\middle| Y\right]\right)^2\right] \leq E_2^{-\nu 2}.$$

3. Decay Rate of Second Moment - Importance Sampling:

$$\frac{1}{\nu}\, E_1^{-\nu} \leq \left(1 + \frac{\alpha C^T\Sigma^{-1}C}{\nu}\right)^{-\nu/2} \mathbb{E}\left[\exp\left(\frac{3\alpha Y}{2\nu}\, C^T\Sigma^{-1}C\right)\mathbb{I}\left(AW < 2\sqrt{\frac{\alpha Y}{\nu}}\, C\right)\right] \leq E_2^{-\nu}.$$

The proof of these two theorems is shown in Appendix.

6 An Application: Nth-to-Default Probability Estimation

In the application of pricing basket default swaps, it often needs to deal with nth-to-default probability given d-dimensional assets. For example, the evaluation of *default leg* is defined as

The mechanism of a k^{th}-to-default swap resembles that of an insurance. From the protection buyer's viewpoint (the *protection leg*), periodical premium payments are made until some credit events happen; meanwhile, swap issuer needs to compensate the non-recovered part[1] of the reference entities' notional amounts[2] (the *default leg*). Pricing a BDS is equivalent to determining the fair premium a BDS buyer needs to pay. Under risk-neutral measure, the fair premium is determined by equating the expected cash flows bilaterally.

Here we introduce some notations and assumptions of our pricing model.

- n: Number of names in one basket, usually 5 or 6.
- T: Terminal date of a BDS contract.
- R_i: Recovery rate of the ith underlying asset.
- M_i: Notional amount, or face value of ith obligator.
- $\triangle_{j-1,\,j}$, $j = 1, 2, \cdots, N$: The time increment $t_j - t_{j-1}$. The market convention is 0.25 year, so we assume quarterly payment.
- h_i: Hazard rate of the ith obligator.
- τ_i: Default time of the ith company.
- $B(0, \tau)$: $\exp\left(-\int_o^\tau r(u)du\right)$, the discount factor, (or price of zero coupon bond) where $r(\cdot)$ denotes risk free interest rate.

In addition, $\mathbb{I}(.)$ stands for the Dirac function and *prem* is the fair premium. The cashflow when k^{th} default takes place (default leg) is

$$DL = \mathbb{E}\left\{(1 - R) \times M \times B(0, \tau) \times \mathbb{I}(\tau < T)\right\}, \qquad (20)$$

where $\mathbb{E}$ is the expectation under risk neutral measure.

On the other hand, the cashflow when obligators do not default (protection leg) is

$$PL = \mathbb{E}\left\{\sum_{j=1}^{N} \triangle_{j-1,j} \times M \times prem \times B(0, t_j) \times \mathbb{I}(\tau \geq t_j)\right\}. \qquad (21)$$

We equate the both sides of (20) and (21) and formulate the unknown premium:

$$prem = \frac{\mathbb{E}\left\{(1 - R) \times B(0, \tau) \times \mathbb{I}(\tau < T)\right\}}{\mathbb{E}\left\{\sum_{j=1}^{N} \triangle_{j-1,j} \times B(0, t_j) \times \mathbb{I}(\tau > t_j)\right\}}. \qquad (22)$$

[1] Often called Loss Given Default (LGD), which might be time-varying and correlated to some macroeconomic conditions.

[2] In homogeneous basket, these notional amounts are assumed to be identical.

To be more precise, we take accrued interests into considerations, modifying (21) by:

$$PL^{acc} = \mathbb{E}\left\{ \sum_{j=1}^{N} (\frac{\tau - t_{j-1}}{t_j - t_{j-1}} \triangle_{j-1,\,j}) \times M \times s \times B(0,\tau) \times \mathbb{I}(t_{j-1} < \tau \leq t_j) \right\}, \tag{23}$$

when defaults happen between two payment dates.

Next, in (20) to estimate default leg value, we assume the risk-free interest rate is zero and it is for simplicity. Thus the value of default leg is reduced to the estimation of the nth-to-default probability:

$$P(k) = \mathbb{P}\left(\tau_k < T\right), \tag{24}$$

where $\tau(k)$ is the kth order statistic of default times $\{\tau_1, \tau_2, \cdots, \tau_d\}$. Note that since for each $i \in \{1, \cdots, d\}, \tau_i = F^{-1}\left(\Phi(X_i)\right)$ and functions F and Φ are monotone, the ith order statistic in τs is preserved by the ith order statistic in X. Therefore, $\mathbb{P}\left(\tau(k) < T\right) = \mathbb{P}\left(\tau_X(k) < c\right)$.

Our importance sampling algorithms to estimate the probability of order statistics for multivariate normal and Student's t random variables are simply easy generalization of previously proposed importance sampling algorithms. Next, we present numerical results.

6.1 Numerical Comparison

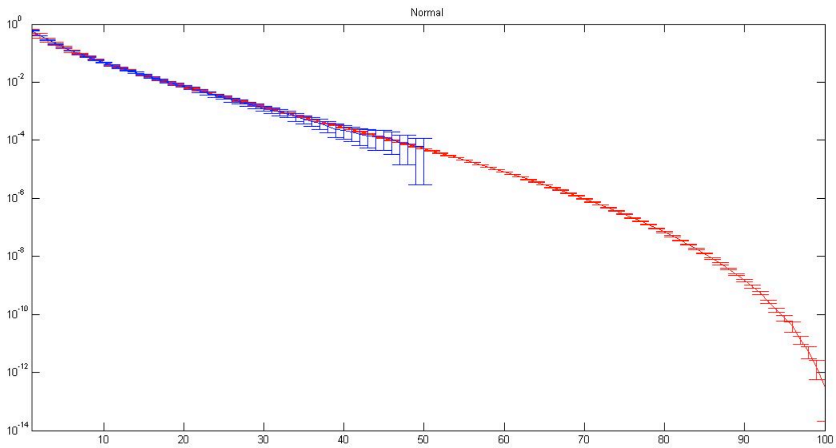

Figure 1: Estimation for nth-to-default Probability of Multivariate Normal given $\rho = 0.5$ and the total number of simulation is $m = 100000$.

In Figure 1 and Figure 2, we estimate a set of nth-to-default probabilities for $n \in \{1, 2, \cdots, d\}$ under Gaussian and Student't distributions, respectively. The model parameters and the total number of simulation are chosen the same as in Sections 3.1.1 and 3.2.1, respectively. Note that our importance sampling algorithms to estimate these order statistic probabilities are simple generalization of previously proposed algorithms; while there is no other (commercial) algorithms developed to estimate these quantities, to the best of our knowledge.

From Figures 1 and 2, it is observed that the order-statistic probabilities fluctuates more under Student's t copula than the Gaussian copula. This is because the Chi-square random variable provides extra variability so that the estimator is more flexible than the one under Gaussian copula.

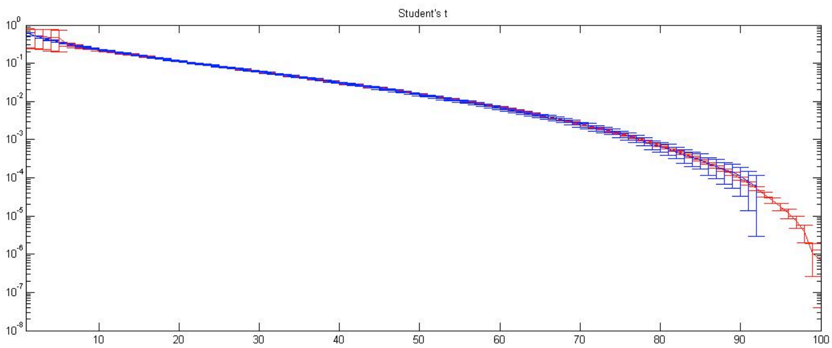

Figure 2: Estimation for nth-to-default Probability of Multivariate Student t given $\rho = 0.5$, $\nu = 3$ and $m = 200000$.

7 Conclusion

Motivated from Gaussian and Student's t copula models in credit risk, importance sampling algorithms are proposed to estimate lower tail probabilities of multivariate normal and Student's t. These algorithms are stochastic alternatives to other deterministic based algorithms. For instance, see Matlab codes *mvncdf.m* and *mvtcdf.m*. Moreover, we show that the proposed algorithms are asymptotically optimal by means of the large deviation principle and a truncation argument. The flexibility of these stochastic algorithms is demonstrated by a generalization to treat the problem of estimating order statistics (nth-to-default) of a credit portfolio.

A Proof of Theorem 1 and Theorem 2

separated into the following results in this section. Recall that using the technique of large deviation, we know that

$$\lim_{\alpha \to \infty} \frac{1}{\alpha} \log \mathbb{E}\left[\mathbb{I}(AW < \sqrt{\alpha}\ C)\right] = -\frac{1}{2}\ C^T \Sigma^{-1} C.$$

This implies that for $\varepsilon > 0$, there exists $\Delta > 0$ such that for $\alpha > \Delta$,

$$\exp\left(-\left(\frac{1}{2}\,C^T\Sigma^{-1}C + \varepsilon\right)\alpha\right) < \mathbb{E}\left[\mathbb{I}(AW < \sqrt{\alpha}\,C)\right] < \exp\left(-\left(\frac{1}{2}\,C^T\Sigma^{-1}C - \varepsilon\right)\alpha\right). \tag{25}$$

Lemma 3. *Let Y be a Chi-square distributed random variable with degree of freedom ν and let α and F be constant independent of ν.*

1. *If $F < \alpha/3$, then for large enough α,*

$$\mathbb{P}\left(Y \leq \frac{\nu F}{\alpha}\right) \sim \alpha^{-\nu/2}.$$

2. *If $F < 2$,*

$$\mathbb{P}\left(Y \leq F\right) \sim \frac{1}{\nu}\left(\frac{2}{F}\right)^{-\nu/2}.$$

Proof: It is known that the cumulative distribution function of Y is given by

$$\mathbb{P}(Y \leq D) = K_2 \int_0^{D/2} t^{\frac{\nu}{2}-1} e^{-t}\, dt. \tag{26}$$

1. Consider $D = \nu F/\alpha$.

   (a) $\nu = 2$: ($\sim$) It is easy to see that if α is large,

   $$\int_0^{D/2} e^{-t}\, dt = (1 - e^{-D/2}) \sim \frac{D}{2} = \frac{\nu F}{2}\,\alpha^{-1},$$

   (b) "$\leq$": For the case $\nu \geq 3$, since the function $t^{\frac{\nu}{2}-1}e^{-t}$ is increasing on $(0, D/2)$,

   $$\int_0^{D/2} t^{\frac{\nu}{2}-1} e^{-t}\, dt \leq \left(\frac{D}{2}\right)^{\nu/2} e^{-D/2} \leq \left(\frac{\nu F}{2}\right)^{\nu/2} \alpha^{-\nu/2}.$$

   For the case $\nu = 1$, using the integration by part, we get

   $$\int_0^{D/2} t^{-1/2} e^{-t}\, dt = (2D)^{1/2} e^{-D/2} + 2\int_0^{D/2} t^{1/2} e^{-t}\, dt \leq K\alpha^{-1/2},$$

   when α is large enough.

   (c) "$\geq$": For the case $\nu = 1$, since the function $t^{-1/2}e^{-t}$ is decreasing in t and $e^{-D/2} >$ some positive constant,

   $$\int_0^{D/2} t^{-1/2} e^{-t}\, dt \geq \frac{D}{2} \cdot \left(\frac{D}{2}\right)^{-1/2} e^{-D/2} \geq k\alpha^{-1/2}.$$

   For the case $\nu \geq 3$, by integration by parts, we have

   $$\int_0^{D/2} t^{\frac{\nu}{2}-1} e^{-t}\, dt \geq \frac{2}{\nu}\left(\frac{D}{2}\right)^{\nu/2} e^{-D/2} \geq k\alpha^{-\nu/2}.$$

   Combing the above three cases, we obtain the desired result.

2. Since $\mathbb{P}(Y \le F)$ is monotone in ν if $F < 2$, we consider here only the case where $\nu = 2m$. In this case,

$$\int_0^{F/2} t^{m-1} e^{-t}\, dt = (m-1)! \left[1 - e^{-F/2} \sum_{i=0}^{m-1} \frac{1}{i!} \left(\frac{F}{2}\right)^i \right].$$

Using the Taylor expansion $e^x - \frac{e^{F/2} x^m}{m!} \le \sum_{i=0}^{m-1} \frac{x^i}{i!} \le e^x - \frac{x^m}{m!}$ for $x \in [0, F/2]$. Thus, we have

$$\frac{e^{-F/2}}{m} \left(\frac{F}{2}\right)^m \le \int_0^{F/2} t^{m-1} e^{-t}\, dt \le \frac{1}{m} \left(\frac{F}{2}\right)^m,$$

which leads to our results. For the general case, using the monotonicity property of $\int_0^{F/2} t^{m-1} e^{-t}\, dt$ in ν.

A.1 Proof of Theorem 1

1. For fixed ν and constant $F > \Delta$,

$$\begin{aligned} & \mathbb{E}\left[\mathbb{I}\left(\frac{AW}{\sqrt{Y/\nu}} < \sqrt{\alpha}\, C \right) \right] \\ = & \mathbb{E}\left[\mathbb{E}\left[\mathbb{I}\left(AW < \sqrt{\alpha Y/\nu}\, C \right) \Big| Y \right] \mathbb{I}\left(Y > \frac{\nu F}{\alpha} \right) \right] \\ & + \mathbb{E}\left[\mathbb{E}\left[\mathbb{I}\left(AW < \sqrt{\alpha Y/\nu}\, C \right) \Big| Y \right] \mathbb{I}\left(Y \le \frac{\nu F}{\alpha} \right) \right]. \end{aligned}$$

On set set $\{Y > \nu F/\alpha\}$, we have $\frac{\alpha Y}{\nu} > F > \Delta$. Since W is a Gaussian vector, by (25) and the moment generating function of chi-square distribution

$$\begin{aligned} & \mathbb{E}\left[\mathbb{E}\left[\mathbb{I}\left(AW < \sqrt{\alpha Y/\nu}\, C \right) \Big| Y \right] \mathbb{I}\left(Y > \frac{\nu F}{\alpha} \right) \right] \\ \le & \mathbb{E}\left[\exp\left(-\left(\frac{1}{2}\, C^T \Sigma^{-1} C - \varepsilon \right) \frac{\alpha Y}{\nu} \right) \mathbb{I}\left(Y > \frac{\nu F}{\alpha} \right) \right] \\ \le & \left(1 + \left(C^T \Sigma^{-1} C - 2\varepsilon \right) \frac{\alpha}{\nu} \right)^{-\nu/2} \sim \alpha^{-\nu/2}, \end{aligned}$$

when α is large enough. Moreover, it is not difficult to see that

$$\begin{aligned} \mathbb{P}\left(Y \le \frac{\nu F}{\alpha} \right) & \ge \mathbb{E}\left[\mathbb{E}\left[\mathbb{I}\left(AW < \sqrt{\alpha Y/\nu}\, C \right) \Big| Y \right] \mathbb{I}\left(Y \le \frac{\nu F}{\alpha} \right) \right] \\ & \ge \mathbb{E}\left[\mathbb{I}\left(AW < \sqrt{\alpha D/\nu}\, C \right) \mathbb{I}\left(Y \le \frac{\nu F}{\alpha} \right) \right] = \text{constant}\ \mathbb{P}\left(Y \le \frac{\nu F}{\alpha} \right) \end{aligned}$$

due to the independence of Z and Y. By Lemma 3

$$\mathbb{E}\left[\mathbb{E}\left[\mathbb{I}\left(AW < \sqrt{\alpha Y/\nu}\, C \right) \Big| Y \right] \mathbb{I}\left(Y \le \frac{\nu F}{\alpha} \right) \right] \sim \mathbb{P}\left(Y \le \frac{\nu F}{\alpha} \right) \sim \alpha^{-\nu/2}.$$

Hence, we get the desired result.

2. Consider

$$\begin{aligned}
&\mathbb{E}\left[\left(\mathbb{E}\left[\mathbb{I}\left(AW<\sqrt{\frac{\alpha Y}{\nu}}\,C\right)\middle|Y\right]\right)^2\right]\\
=\;&\mathbb{E}\left[\left(\mathbb{E}\left[\mathbb{I}\left(AW<\sqrt{\frac{\alpha Y}{\nu}}\,C\right)\middle|Y\right]\right)^2\mathbb{I}\left(Y\le\frac{\nu F}{\alpha}\right)\right]\\
&+\mathbb{E}\left[\left(\mathbb{E}\left[\mathbb{I}\left(AW<\sqrt{\frac{\alpha Y}{\nu}}\,C\right)\middle|Y\right]\right)^2\mathbb{I}\left(Y>\frac{\nu F}{\alpha}\right)\right]\\
=\;& I+\mathit{II},\text{ say.}
\end{aligned}$$

A similar argument as in the first part of the proof of Proposition **??**, we get

$$I\sim\mathbb{P}\left(Y\le\frac{\nu F}{\alpha}\right)\sim\alpha^{-\nu/2}.$$

Moreover, we have

$$0\le\mathit{II}\le\mathbb{E}\left[\left(\exp\left(-\left(\frac{1}{2}\,C^T\Sigma^{-1}C-\varepsilon\right)\,\frac{\alpha Y}{\nu}\right)\right)^2\mathbb{I}(Y>\frac{\nu F}{\alpha})\right]\le\left(1+2(C^T\Sigma^{-1}C-2\varepsilon)\,\frac{\alpha}{\nu}\right)^{-\nu/2}.$$

Hence, if α is large enough, we get that $0\le\mathit{II}\le\bar{K}\alpha^{-\nu/2}$. Thus, we can get

$$\mathbb{E}\left[\left(\mathbb{E}\left[\mathbb{I}\left(X<\sqrt{\frac{\alpha Y}{\nu}}\,C\right)\middle|Y\right]\right)^2\right]\sim\alpha^{-\nu/2}.$$

3.

$$\begin{aligned}
&\mathbb{E}\left[\exp\left(\frac{3\alpha Y}{2\nu}\,C^T\Sigma^{-1}C\right)\mathbb{I}\left(AW<2\sqrt{\frac{\alpha Y}{\nu}}\,C\right)\right]\\
=\;&\mathbb{E}\left[\exp\left(\frac{3\alpha Y}{2\nu}\,C^T\Sigma^{-1}C\right)\mathbb{E}\left[\mathbb{I}\left(AW<2\sqrt{\frac{\alpha Y}{\nu}}\,C\right)\middle|Y\right]\mathbb{I}\left(Y>\frac{\nu F}{\alpha}\right)\right]\\
+\;&\quad+\mathbb{E}\left[\exp\left(\frac{3\alpha Y}{2\nu}\,C^T\Sigma^{-1}C\right)\mathbb{E}\left[\mathbb{I}\left(AW<2\sqrt{\frac{\alpha Y}{\nu}}\,C\right)\middle|Y\right]\mathbb{I}\left(Y\le\frac{\nu F}{\alpha}\right)\right]\\
=\;&I+II,\text{ say.}
\end{aligned}$$

Due to the independence of Z and Y and the large deviation, we see that

$$\begin{aligned}
I\;&\le\;\mathbb{E}\left[\exp\left(\frac{3\alpha Y}{2\nu}\,C^T\Sigma^{-1}C\right)\exp\left(-\left(\frac{1}{2}\,C^T\Sigma^{-1}C-\varepsilon\right)\frac{4\alpha Y}{\nu}\right)\right]\\
&=\;\left(1+(C^T\Sigma^{-1}C-8\varepsilon)\,\frac{\alpha}{\nu}\right)^{-\nu/2}.
\end{aligned}$$

Thus,

$$\left(1+\frac{\alpha C^T\Sigma^{-1}C}{\nu}\right)^{-\nu/2}I\le\left(1+(C^T\Sigma^{-1}C-8\varepsilon)\,\frac{\alpha}{\nu}\right)^{-\nu}\sim\alpha^{-\nu}$$

Moreover, using a similar argument as in Proposition **??**,

$$II\le\exp\left(\frac{3F}{2}\,C^T\Sigma^{-1}C\right)\mathbb{P}\left(Y\le\frac{\nu F}{\alpha}\right)\sim\alpha^{-\nu/2},$$

which implies that

$$\left(1+\frac{\alpha C^T\Sigma^{-1}C}{\nu}\right)^{-\nu/2} II \leq K\alpha^{-\nu}.$$

Hence, the value (**??**) has upper bound $K\alpha^{-\nu}$. Furthermore, since the mean of joint default is approximated to $\alpha^{-\nu/2}$, we can get the lower bound is also $K\alpha^{-\nu}$.

A.2 Proof of Theorem 2

1. For $k-1<\frac{\alpha}{\nu}<k$ and a constant F,

$$\begin{aligned}&\mathbb{E}\left[\mathbb{I}\left(\frac{AW}{\sqrt{Y/\nu}}<\sqrt{\alpha}\;C\right)\right]\\ =\;&\mathbb{E}\left[\mathbb{E}\left[\mathbb{I}\left(AW<\sqrt{\alpha Y/\nu}\;C\right)\Big|\,Y\right]\mathbb{I}\left(Y>F\right)\right]+\mathbb{E}\left[\mathbb{E}\left[\mathbb{I}\left(AW<\sqrt{\alpha Y/\nu}\;C\right)\Big|\,Y\right]\mathbb{I}\left(Y\leq F\right)\right].\end{aligned}$$

Using a similar argument, we have

$$\mathbb{E}\left[\mathbb{E}\left[\mathbb{I}\left(AW<\sqrt{\alpha Y/\nu}\;C\right)\Big|\,Y\right]\mathbb{I}\left(Y>F\right)\right]\leq\left(1+\left(\;C^T\Sigma^{-1}C-2\varepsilon\right)\frac{\alpha}{\nu}\right)^{-\nu/2}\leq M_1^{-\nu/2},$$

and

$$\mathbb{E}\left[\mathbb{E}\left[\mathbb{I}\left(AW<\sqrt{\alpha Y/\nu}\;C\right)\Big|\,Y\right]\mathbb{I}\left(Y\leq F\right)\right]\sim\mathbf{P}(Y\leq F)\sim\frac{1}{\nu}\;M_2^{-\nu/2}.$$

This leads to our results.

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國科會補助計畫衍生研發成果推廣資料表

日期:2012/09/27

國科會補助計畫	計畫名稱: 連續時間 filtration 的正交分解
	計畫主持人: 吳慶堂
	計畫編號: 100-2115-M-009-002- 學門領域: 隨機分析與統計物理

無研發成果推廣資料

100 年度專題研究計畫研究成果彙整表

計畫主持人：吳慶堂	計畫編號：100-2115-M-009-002-

計畫名稱：連續時間 filtration 的正交分解

成果項目			量化：實際已達成數（被接受或已發表）	量化：預期總達成數(含實際已達成數)	量化：本計畫實際貢獻百分比	單位	備註（質化說明：如數個計畫共同成果、成果列為該期刊之封面故事...等）
國內	論文著作	期刊論文	1	0	100%	篇	
		研究報告/技術報告	0	0	100%		
		研討會論文	0	0	100%		
		專書	0	0	100%		
	專利	申請中件數	0	0	100%	件	
		已獲得件數	0	0	100%		
	技術移轉	件數	0	0	100%	件	
		權利金	0	0	100%	千元	
	參與計畫人力（本國籍）	碩士生	0	0	100%	人次	
		博士生	0	0	100%		
		博士後研究員	0	0	100%		
		專任助理	0	0	100%		
國外	論文著作	期刊論文	0	0	100%	篇	
		研究報告/技術報告	0	0	100%		
		研討會論文	0	0	100%		
		專書	0	0	100%	章/本	
	專利	申請中件數	0	0	100%	件	
		已獲得件數	0	0	100%		
	技術移轉	件數	0	0	100%	件	
		權利金	0	0	100%	千元	
	參與計畫人力（外國籍）	碩士生	0	0	100%	人次	
		博士生	0	0	100%		
		博士後研究員	0	0	100%		
		專任助理	0	0	100%		

其他成果 (無法以量化表達之成果如辦理學術活動、獲得獎項、重要國際合作、研究成果國際影響力及其他協助產業技術發展之具體效益事項等，請以文字敘述填列。)	無

	成果項目	量化	名稱或內容性質簡述
科教處計畫加填項目	測驗工具(含質性與量性)	0	
	課程/模組	0	
	電腦及網路系統或工具	0	
	教材	0	
	舉辦之活動/競賽	0	
	研討會/工作坊	0	
	電子報、網站	0	
	計畫成果推廣之參與（閱聽）人數	0	

國科會補助專題研究計畫成果報告自評表

請就研究內容與原計畫相符程度、達成預期目標情況、研究成果之學術或應用價值（簡要敘述成果所代表之意義、價值、影響或進一步發展之可能性）、是否適合在學術期刊發表或申請專利、主要發現或其他有關價值等，作一綜合評估。

1. 請就研究內容與原計畫相符程度、達成預期目標情況作一綜合評估

   □達成目標

   ■未達成目標（請說明，以 100 字為限）

   　　□實驗失敗

   　　□因故實驗中斷

   　　■其他原因

   說明：

   因 Professor Yor 身體狀況不佳, Yor 與 Strook 之文章一直未完成, 雖然已得到一些小結果, 但尚未到達能發表及形成成果報告的程度. 因此在這段時間我便開始從事另一方面主題的研究. 因此僅能附上這段時間的另一些研究成果.

2. 研究成果在學術期刊發表或申請專利等情形：

   論文：□已發表 □未發表之文稿 ■撰寫中 □無

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   技轉：□已技轉 □洽談中 ■無

   其他：（以 100 字為限）

   附上這段時間正在撰寫中的論文.

3. 請依學術成就、技術創新、社會影響等方面，評估研究成果之學術或應用價值（簡要敘述成果所代表之意義、價值、影響或進一步發展之可能性）（以 500 字為限）

   In credit risk modeling, Gaussian and Student's t variates arise primarily from the copula method to retain certain correlation structures among defaultable assets. We propose efficient importance sampling algorithms to estimate lower tail probabilities of these two variates in any finite dimension. Variances of importance sampling estimators are shown asymptotically optimal by means of the large deviation theory and a truncation argument. Numerical comparisons with commercial codes, such as mvncdf.m and mvtcdf.m in Matlab, demonstrate robustness and efficiency of our proposed algorithms. Moreover, the flexibility of these algorithms can be seen from an application of probability estimation for the nth-to-default, i.e., the nth order statistic, given a credit portfolio.